Conformally Einstein and Bach-flat four-dimensional homogeneous manifolds*

E. Calviño-Louzao[a], X. García-Martínez[b,c], E. García-Río[d,*], I. Gutiérrez-Rodríguez[d], R. Vázquez-Lorenzo[e]

[a]*Consellería de Cultura, Educación e Ordenación Universitaria, Edificio Administrativo San Caetano, 15781 Santiago de Compostela, Spain*
[b]*Departamento de Matemáticas, Esc. Sup. de Enx. Informática, Campus de Ourense, Universidade de Vigo, 32004, Ourense, Spain*
[c]*Faculty of Engineering, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussel, Belgium*
[d]*Faculty of Mathematics, University of Santiago de Compostela, 15782 Santiago de Compostela, Spain*
[e]*Department of Mathematics, IES de Ribadeo Dionisio Gamallo, 27700 Ribadeo, Spain*

Abstract

Homogeneous conformally Einstein manifolds are classified in dimension four. As a consequence we show that any homogeneous strictly Bach-flat four-dimensional manifold is homothetic to one of the examples constructed by Abbena, Garbiero and Salamon in [1].

Résumé

Dans cet article, nous classifions les variétés homogènes conformément d'Einstein de dimension quatre. En conséquence, nous montrons que toute variété homogène strictement Bach-plate de dimension quatre est homothétique à l'un des exemples construits par Abbena, Garbiero et Salamon dans [1].

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1. Introduction

Einstein metrics, being critical for the Hilbert functional, are central in Riemannian geometry. Jensen [24] classified four-dimensional homogeneous Einstein manifolds, showing that they are locally symmetric and thus very rigid. A more general question is to classify homogeneous four-manifolds which have an Einstein representative in their conformal class, i.e., to classify conformally Einstein homogeneous four-manifolds. Locally conformally flat metrics are trivially conformally Einstein. The conformal flatness assumption is again very rigid in the homogeneous setting, as Takagi showed that it leads to locally symmetric spaces [34]. Conversely, a four-dimensional locally symmetric conformally Einstein metric is either Einstein or locally conformally flat (cf. Lemma 2.3). Hence, aiming to describe conformally Einstein homogeneous metrics we focus on the non-symmetric ones.

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*Corresponding author

Email addresses: `estebcl@edu.xunta.es` (E. Calviño-Louzao), `xabier.garcia.martinez@uvigo.gal` (X. García-Martínez), `eduardo.garcia.rio@usc.es` (E. García-Río), `ixcheldzohara.gutierrez@usc.es` (I. Gutiérrez-Rodríguez), `ravazlor@edu.xunta.es` (R. Vázquez-Lorenzo)

Theorem 1.1. *Let (M,g) be a four-dimensional complete and simply connected conformally Einstein homogeneous manifold. Then (M,g) is locally symmetric or otherwise it is homothetic to one of the Lie groups determined by the following solvable Lie algebras:*

(i) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ given by

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = \frac{1}{4}e_2 + \alpha e_3, \quad [e_4, e_3] = -\alpha e_2 + \frac{1}{4}e_3 \,.$$

(ii) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{h}^3$ given by

$$[e_1, e_2] = e_3 \quad [e_4, e_1] = e_1 - \alpha e_2, \quad [e_4, e_2] = \alpha e_1 + e_2, \quad [e_4, e_3] = 2e_3 \,.$$

(iii) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ given by

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = (\alpha+1)^2\, e_2, \quad [e_4, e_3] = \alpha^2\, e_3, \qquad \alpha > 1 \,.$$

Here $\{e_1, \dots, e_4\}$ is an orthonormal basis. Moreover, the Lie groups $(G_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle)$ in Assertion (i) are half conformally flat.

Following the notation in [3], the underlying Lie algebras in Theorem 1.1 are $\mathfrak{r}'_{4,1,\frac{1}{4\alpha}}$ if $\alpha \neq 0$ or $\mathfrak{r}_{4,\frac{1}{4},\frac{1}{4}}$ if $\alpha = 0$ in Assertion (i), $\mathfrak{d}'_{4,\frac{1}{\alpha}}$ if $\alpha \neq 0$ or $\mathfrak{d}_{4,\frac{1}{2}}$ if $\alpha = 0$ in Assertion (ii) and $\mathfrak{r}_{4,(\alpha+1)^2,\alpha^2}$ in Assertion (iii).

Among the different quadratic curvature functionals the one given by the L^2-norm of the Weyl tensor, $\mathcal{F}_W : g \mapsto \mathcal{F}_W(g) = \int_M \|W(g)\|^2$, is conformally invariant. Bach-flat metrics, being the critical metrics of $\mathcal{F}_W$, play an important role in general relativity and geometry. Locally conformally flat metrics are trivially Bach-flat. Moreover, the Hirzebruch signature formula, $48\pi^2\tau[M] = \int_M \{\|W^+\|^2 - \|W^-\|^2\}$, shows that half conformally flat metrics are critical for $\mathcal{F}_W$ and thus Bach-flat. Furthermore, the conformal invariance of $\mathcal{F}_W$ shows that conformally Einstein metrics are Bach-flat since Einstein metrics are so. Bach-flatness, a condition already investigated in conformally invariant gravitational theories, is the most natural generalization of Einstein metrics.

Four-dimensional locally symmetric Bach-flat metrics are either Einstein or locally conformally flat (cf. Lemma 2.3). The existence of left-invariant Riemannian metrics with zero Bach tensor which are neither conformally Einstein nor half conformally flat was established in [1]. We show that the examples constructed by Abbena, Garbiero and Salamon are the only possible ones within the framework of four-dimensional homogeneous manifolds.

Theorem 1.2. *Let (M,g) be a four-dimensional complete and simply connected strictly Bach-flat homogeneous manifold. Then (M,g) is homothetic to one of the Lie groups determined by the following solvable Lie algebras:*

(i) The Lie algebra $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{e}(1,1)$ given by

$$[e_2, e_3] = e_1, \quad [e_1, e_3] = (2+\sqrt{3})\, e_2, \quad [e_4, e_1] = \sqrt{6+3\sqrt{3}}\, e_1, \quad [e_4, e_2] = \sqrt{6+3\sqrt{3}}\, e_2.$$

(ii) The Lie algebra $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{h}_3$ given by

$$[e_1, e_2] = e_3, \quad [e_4, e_1] = \frac{1}{4}\sqrt{7-3\sqrt{5}}\, e_1, \quad [e_2, e_4] = \frac{1}{4}\sqrt{7+3\sqrt{5}}\, e_2, \quad [e_3, e_4] = \frac{\sqrt{5}}{2\sqrt{2}}\, e_3,$$

where $\{e_1, \dots, e_4\}$ is an orthonormal basis.

The underlying Lie algebras in Theorem 1.2 are $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$ in case (i) and $\mathfrak{d}_{4,\mu}$ with $\mu = \frac{1}{10}(5-3\sqrt{5})$ in case (ii), following again the notation in [3].

Ricci solitons, being self-similar solutions of the Ricci flow, provide another important generalization of Einstein metrics. Complete Bach-flat gradient Ricci solitons are locally conformally flat in the shrinking and steady cases [11, 12] (see also [15]). Moreover, since homogeneous gradient Ricci solitons are rigid (see [33]), we focus in the generic case where the Ricci soliton vector field is not a gradient. The generic situation supports homogeneous Bach-flat Ricci solitons which are expanding, as follows.

Corollary 1.3. *Let (M, g) be a four-dimensional complete and simply connected Bach-flat homogeneous Ricci soliton. Then (M, g) is Einstein, a rigid gradient Ricci soliton $N(c) \times \mathbb{R}^k$, or homothetic to one of the algebraic Ricci solitons determined by the following solvable Lie algebras:*

(i) The Lie algebra given at Theorem 1.1(i).

(ii) The Lie algebra given at Theorem 1.2(ii).

(iii) The Lie algebra given at Theorem 1.1(iii).

The paper is organized as follows. In Section 2 we recall a result of Bérard-Bergery (see Theorem 2.1) which asserts that a complete and simply connected homogeneous four-manifold is either symmetric or a Lie group. Locally symmetric Bach-flat four-manifolds are shown to be either Einstein or locally conformally flat (cf. Lemma 2.3). Hence the analysis of the Bach-flat condition is considered separately for the different four-dimensional Lie groups through Sections 3–6. The components of the Bach tensor give polynomials in the corresponding structure constants. Therefore, determining the Bach-flat Lie groups amounts to solving some rather complicated polynomial systems. We make use of Gröbner bases theory previously introduced in Section 2.5. Finally, the proofs of Theorem 1.1, Theorem 1.2 and Corollary 1.3 are given in Section 7, as well as some structure results.

2. Preliminaries

2.1. Four-dimensional homogeneous manifolds

Bérard-Bergery showed in [8] that any four-dimensional Riemannian homogeneous space is either symmetric or a Lie group. More precisely:

Theorem 2.1. [8] *Let (M, g) be a four-dimensional complete and simply connected homogeneous manifold. Then either (M, g) is locally symmetric or it is isometric to a Lie group with a left-invariant metric.*

In particular, either M is one of the groups $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$, $SU(2) \times \mathbb{R}$ or it is a solvable Lie group. Four-dimensional solvable Lie algebras are obtained as extensions of the three-dimensional unimodular Lie algebras: the abelian Lie algebra $\mathfrak{r}^3$, the Heisenberg algebra $\mathfrak{h}_3$, the Poincaré algebra $\mathfrak{e}(1,1)$ and the Euclidean algebra $\mathfrak{e}(2)$. Moreover, the solvable and simply connected four-dimensional Lie groups are the following:

(a) The non-trivial semi-direct products $\mathbb{R} \ltimes E(2)$ and $\mathbb{R} \ltimes E(1,1)$.
(b) The semi-direct products $\mathbb{R} \ltimes \mathbb{R}^3$.
(c) The non-nilpotent semi-direct products $\mathbb{R} \ltimes H^3$, where H^3 is the Heisenberg group.

2.2. Bach-flat metrics

Let (M, g) be an n-dimensional Riemannian manifold and let W be the Weyl conformal curvature tensor. Then the *Bach tensor* is defined by

$$\mathfrak{B} = \operatorname{div}_1 \operatorname{div}_4 W + \frac{n-3}{n-2} W[\rho], \tag{1}$$

where $W[\rho](X, Y) = \sum_{ij} W(E_i, X, Y, E_j)\rho(E_i, E_j)$ for a local orthonormal frame $\{E_i\}$, ρ denotes the Ricci tensor and div denotes the divergence operator. The Bach tensor, introduced in [6] to study conformal relativity, is trace-free and is a conformal density in dimension four. On four-dimensional compact manifolds, Bach-flat metrics are precisely critical metrics of the conformally invariant functional $\mathcal{F}_W(g) = \int_M \|W(g)\|^2$ of the Weyl curvature. Locally conformally flat metrics and half conformally flat metrics are Bach-flat since they are extremal values of the L^2-norm of the Weyl curvature. Einstein metrics and conformally Einstein metrics are also Bach-flat. The Bach tensor is also the gradient of some equivalent functionals in four-dimensional geometry, like the one given by the $\mathcal{Q}$-curvature (see [29] for more information).

We say that (M, g) is *strictly Bach-flat* if the Bach tensor vanishes and (M, g) is neither half conformally flat nor conformally Einstein.

2.2.1. Half conformally flat homogeneous manifolds

A four-dimensional manifold (M, g) is said to be *half conformally flat* if it is either *self-dual* (i.e., $W^- = 0$) or *anti-self-dual* (i.e., $W^+ = 0$). Here $W^\pm = \frac{1}{2}(W \pm *W)$ are the self-dual and anti-self-dual parts of the Weyl curvature tensor, where $*\colon \Lambda^2 \to \Lambda^2$ is the Hodge operator inducing a decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. Since any half conformally flat manifold is Bach-flat, Lemma 2.3 shows that any locally symmetric half conformally flat manifold is either Einstein or locally conformally flat. Based on Theorem 2.1, De Smedt and Salamon [17] addressed the classification of half conformally flat homogeneous manifolds as follows.

Theorem 2.2. [17] *A four-dimensional homogeneous manifold is strictly anti-self-dual if and only if it is a complex space-form or a simply connected Lie group G_α corresponding to the solvable Lie algebra $\mathfrak{g}_\alpha$ given by*

$$[e_1, e_2] = e_2 - \alpha e_3, \quad [e_1, e_3] = \alpha e_2 + e_3, \quad [e_1, e_4] = 2e_4, \quad [e_2, e_3] = -e_4, \tag{2}$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal basis.

2.2.2. Locally symmetric Bach-flat metrics

Four-dimensional homogeneous Einstein manifolds are locally symmetric [24]. Furthermore, any locally conformally flat homogeneous manifold is locally symmetric [34].

Lemma 2.3. *A four-dimensional locally symmetric Bach-flat manifold is Einstein or locally conformally flat.*

Proof. Let (M^4, g) be locally symmetric. Then it is an Einstein manifold or it is locally a product of the form $\mathbb{R} \times N^3(c)$, $\mathbb{R}^2 \times N^2(c)$ or $N_1^2(c_1) \times N_2^2(c_2)$, where $N^k(c)$ is a k-dimensional manifold of constant curvature c. In the case $\mathbb{R} \times N^3(c)$, (M, g) is locally conformally flat since $N^3(c)$ is of constant curvature. An explicit calculation of the Bach tensor shows that $\mathbb{R}^2 \times N^2(c)$, where $N^2(c)$ is a surface of constant curvature, is Bach-flat if and only if $c = 0$, thus (M, g) being flat. Finally, the Bach tensor of $N_1^2(c_1) \times N_2^2(c_2)$ vanishes if and only if $c_1^2 - c_2^2 = 0$, thus leading to locally conformal flatness ($c_1 = -c_2$) or to an Einstein manifold ($c_1 = c_2$).

2.3. Conformally Einstein manifolds

A Riemannian manifold (M, g) is *(locally) conformally Einstein* if every point $p \in M$ has an open neighborhood $\mathcal{U}$ and a positive smooth function φ defined on $\mathcal{U}$ such that $(\mathcal{U}, \widetilde{g} = \varphi^{-2} g)$ is Einstein. Brinkmann [10] showed that (M^n, g) is conformally Einstein if and only if the equation

$$(n-2) \operatorname{Hes}_\varphi + \varphi\, \rho - \frac{1}{n}\{(n-2)\Delta\varphi + \varphi\, \tau\} g = 0 \tag{3}$$

has a positive solution. Besides its apparent simplicity, the integration of the conformally Einstein equation is surprisingly difficult (see, for example [26]).

Any two-dimensional manifold is conformally Einstein, and three-dimensional manifolds are conformally Einstein if and only if they are locally conformally flat. The situation is more subtle in dimensions $n \geq 4$.

Let $\mathfrak{S} = \rho - \frac{\tau}{2(n-1)}\, g$ denote the Schouten tensor of (M, g). Let $\mathfrak{C}$ be the *Cotton tensor*, $\mathfrak{C}_{ijk} = (\nabla_i \mathfrak{S})_{jk} - (\nabla_j \mathfrak{S})_{ik}$. The Bach and the Cotton tensors of any four-dimensional manifold are related by $\mathfrak{B} =$

$\frac{1}{2}\left(-\operatorname{div}_1 \mathfrak{C} + W[\rho]\right)$ since $\operatorname{div}_4 W = -\frac{n-3}{n-2}\,\mathfrak{C}$. It was shown in [21, 25] that any four-dimensional conformally Einstein manifold satisfies

$$(i)\quad \mathfrak{B} = 0, \qquad\qquad (ii)\quad \mathfrak{C} + W(\cdot,\cdot,\cdot,\nabla\sigma) = 0, \tag{4}$$

where the conformal Einstein metric is given by $\widetilde{g} = e^{2\sigma} g$. Conditions (4)(i)–(ii) above are also sufficient to be conformally Einstein if (M,g) is *weakly-generic* (i.e., the Weyl tensor viewed as a map $TM \to \bigotimes^3 TM$ is injective).

Lemma 2.3 shows that four-dimensional locally symmetric manifolds are conformally Einstein if and only if they are either Einstein or locally conformally flat. Hence, in order to describe conformally Einstein homogeneous four-manifolds one has to analyze the conditions in Equation (4) for left-invariant metrics on Lie groups.

2.4. Algebraic Ricci solitons

A Riemannian manifold (M,g) is a *Ricci soliton* if there is a vector field X on M so that

$$\mathcal{L}_X g + \rho = \lambda g, \tag{5}$$

where $\mathcal{L}$ is the Lie derivative, ρ denotes the Ricci tensor and $\lambda \in \mathbb{R}$. A Ricci soliton (M,g,X) is called *shrinking*, *steady* or *expanding* if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Moreover, a Ricci soliton is called *trivial* if (M,g) is Einstein. If X is a gradient, then Equation (5) becomes $\operatorname{Hes}_f + \rho = \lambda g$ for some potential function f and (M,g,f) is called a *gradient Ricci soliton*. Non-trivial homogeneous gradient Ricci solitons are rigid, i.e., M splits as a product $N \times \mathbb{R}^k$, where N is Einstein and the potential function is given by the projection on the Euclidean factor [33].

Ricci solitons are self-similar solutions of the Ricci flow $\frac{\partial}{\partial t} g(t) = -2\rho(g(t))$, i.e., they are fixed points of the flow up to diffeomorphisms and rescaling. On a Lie group one may consider a stronger condition and search for fixed points of the flow up to automorphisms of the Lie group instead of diffeomorphisms. This observation led Lauret [27] to introduce algebraic Ricci solitons. Let G be a Lie group with Lie algebra $\mathfrak{g}$. A left-invariant metric $\langle\,\cdot\,,\,\cdot\,\rangle$ on G is called an *algebraic Ricci soliton* if

$$\mathfrak{D} = \operatorname{Ric} - \lambda \operatorname{Id} \tag{6}$$

is a derivation of the Lie algebra, i.e., $\mathfrak{D}[X,Y] = [\mathfrak{D}X,Y] + [X,\mathfrak{D}Y]$ for all $X,Y \in \mathfrak{g}$, where Ric denotes the Ricci operator ($\langle \operatorname{Ric} X, Y\rangle = \rho(X,Y)$) and $\lambda \in \mathbb{R}$. Let $\mathfrak{D}$ be a derivation given by Equation (6) and let φ_t denote the one-parameter family of automorphisms determined by $d\varphi_t|_e = \exp \frac{t}{2}\mathfrak{D}$. Then the vector field X given by $X(p) = \frac{d}{dt}\varphi_t(p)|_{t=0}$ satisfies Equation (5), thus defining a Ricci soliton on G. It is important to recognize that both Equations (5) and (6) are invariant under homotheties. Hence, aiming to characterize Bach-flat homogeneous Ricci solitons we shall work modulo homotheties.

Let $\Delta_X u = \Delta u - g(X,\nabla u)$ be the X-Laplacian on a Ricci soliton structure (M,g,X) (see, for example [13]). Then $\frac{1}{2}\Delta_X \tau = \lambda\tau - \|\operatorname{Ric}\|^2$, which shows that a steady Ricci soliton with constant scalar curvature is Ricci-flat, and hence flat in the homogeneous setting [2]. Furthermore, four-dimensional homogeneous shrinking Ricci solitons have bounded curvature and thus they are gradient [31] (see also [5]). Every homogeneous expanding Ricci soliton is necessarily non-compact, and all known non-gradient examples are algebraic Ricci solitons on manifolds isometric to solvable Lie groups with left-invariant metrics [23].

2.5. Gröbner bases

The components of the Bach tensor for a left-invariant metric on a Lie group are polynomials in the structure constants. To obtain a full classification of Bach-flat Lie groups, one needs to solve the corresponding polynomial system of equations. When the system under consideration is simple, it is an elementary problem to find all common roots, but if the number of equations and their degrees increase, it may become a quite unmanageable assignment. There exist however some well-known strategies to approach this kind of problem.

Given a set $\mathcal{S}$ of polynomials $\mathfrak{P}_i \in \mathbb{R}[x_1, \ldots, x_n]$, an n-tuple of real numbers $\vec{a} = (a_1, \ldots, a_n)$ is a solution of $\mathcal{S}$ if and only if $\mathfrak{P}_i(\vec{a}) = 0$ for all i. It is immediate to see that $\vec{a}$ is a solution of $\mathcal{S}$ if and only if it is a solution of $\mathcal{I} = \langle \mathfrak{P}_i \rangle$, the ideal generated by the $\mathfrak{P}_i$. Note that if two sets of polynomials generate the same ideal, the corresponding zero sets must be identical. Therefore, our strategy for solving the rather large polynomial systems consists of obtaining "better" polynomials that belong to the ideals generated by the corresponding polynomial systems.

Let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha \in \mathbb{Z}^n_{\geq 0}$ be a monomial in $\mathbb{R}[x_1, \ldots, x_n]$. Establishing an ordering on $\mathbb{Z}^n_{\geq 0}$ will induce an ordering on the monomials. We are specially interested in the following monomial orderings:

- *Lexicographical Order*: We say that $\alpha >_{lex} \beta$ if in the vector $\alpha - \beta \in \mathbb{Z}^n$, the leftmost non-zero entry is positive.

- *Graded Lexicographical Order*: We say that $\alpha >_{grlex} \beta$ if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and $\alpha >_{lex} \beta$, where $|\alpha| = \sum_i \alpha_i$.

- *Graded Reverse Lexicographical Order*: We say that $\alpha >_{grevlex} \beta$ if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and the rightmost non-zero entry of $\alpha - \beta \in \mathbb{Z}^n$ is negative.

If $\mathfrak{P} = \sum_\alpha a_\alpha x^\alpha$ is a polynomial in $\mathbb{R}[x_1, \ldots, x_n]$, any of the monomial orderings above orders the monomials of $\mathfrak{P}$. The *multidegree* of $\mathfrak{P}$ is the maximum $\alpha \in \mathbb{Z}^n_{\geq 0}$ so that $a_\alpha \neq 0$, where the maximum is taken with respect to the given monomial ordering. The corresponding monomial is called the *leading term* $LT(\mathfrak{P}) = a_\alpha x^\alpha$. Let $\mathcal{I} \subset \mathbb{R}[x_1, \ldots, x_n]$ be a non-zero ideal. Let $LT(\mathcal{I})$ be the set of leading terms of all elements of $\mathcal{I}$ and let $\langle LT(\mathcal{I}) \rangle$ be the ideal generated by the elements of $LT(\mathcal{I})$. It is important to emphasize that if $\mathcal{I} = \langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$, then $\langle LT(\mathcal{I}) \rangle$ may be strictly larger than the ideal $\langle LT(\mathfrak{P}_1), \ldots, LT(\mathfrak{P}_k) \rangle$. A finite subset $\mathcal{G} = \{\mathbf{g}_1, \ldots, \mathbf{g}_\nu\}$ of an ideal $\mathcal{I}$ is said to be a *Gröbner basis* (sometimes also called Gröbner-Shirshov basis) with respect to some monomial order if $\langle LT(\mathbf{g}_1), \ldots, LT(\mathbf{g}_\nu) \rangle = \langle LT(\mathcal{I}) \rangle$.

The Hilbert Basis Theorem guarantees that any non-zero ideal $\mathcal{I} \subset \mathbb{R}[x_1, \ldots, x_n]$ has a Gröbner basis. Furthermore, any Gröbner basis for an ideal $\mathcal{I}$ is a basis of $\mathcal{I}$. Buchberger's algorithm (among others) provides a constructive algorithm to find one such basis. This rather simple notion allows us to have simple algorithmic solutions to different problems.

- The remainder of the division algorithm applied to a polynomial $\mathfrak{P}$ divided by a Gröbner basis $\mathcal{G}$ of an ideal $\mathcal{I}$ is zero if and only if $\mathfrak{P}$ belongs to $\mathcal{I}$, a property that does not necessarily hold if $\mathcal{G}$ is not a Gröbner basis. Therefore, this fact provides an algorithm to check the Ideal Membership Problem.

- As another example, when the set of solutions of a polynomial system is not too large, the calculation of a Gröbner basis with respect the lexicographical order gives rise to elimination theory, simplifying the problem of finding all common roots, thus generalizing the classical Gaussian method of the linear case.

Just as a matter of curiosity, let us mention that Gröbner bases even generalize the simplex method used in mathematical optimization. We refer the interested reader to [16] for more information on the theory of Gröbner bases.

3. Left-invariant metrics on $\mathbb{R}e_4 \ltimes E(1,1)$ and $\mathbb{R}e_4 \ltimes E(2)$

Let $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{g}_3$ be a semi-direct extension of the unimodular Lie algebra $\mathfrak{g}_3 = \mathfrak{e}(1,1)$ or $\mathfrak{g}_3 = \mathfrak{e}(2)$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ and $\langle \cdot, \cdot \rangle_3$ its restriction to $\mathfrak{g}_3$. Following the work of Milnor [30], there exists an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{g}_3$ such that

$$[\mathbf{v}_2, \mathbf{v}_3] = \lambda_1 \mathbf{v}_1, \qquad [\mathbf{v}_3, \mathbf{v}_1] = \lambda_2 \mathbf{v}_2, \qquad [\mathbf{v}_1, \mathbf{v}_2] = 0, \tag{7}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1\lambda_2 \neq 0$. Moreover, the associated Lie group corresponds to $E(2)$ (resp., $E(1,1)$) if $\lambda_1\lambda_2 > 0$ (resp., $\lambda_1\lambda_2 < 0$). The algebra of derivations of $\mathfrak{g}_3$ is given by

$$\operatorname{der}(\mathfrak{g}_3) = \left\{ \begin{pmatrix} b & a & c \\ -\frac{\lambda_2}{\lambda_1}a & b & d \\ 0 & 0 & 0 \end{pmatrix} ;\, a, b, c, d \in \mathbb{R} \right\}.$$

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis of $\mathfrak{g}$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are given by Equation (7) and $\mathfrak{g} = \mathbb{R}\mathbf{v}_4 \oplus \mathfrak{g}_3$. Since $\mathbb{R}\mathbf{v}_4$ needs not to be orthogonal to $\mathfrak{g}_3$, set $k_i = \langle \mathbf{v}_i, \mathbf{v}_4 \rangle$, for $i = 1, 2, 3$. Let $\hat{e}_4 = \mathbf{v}_4 - \sum_i k_i \mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1, \ldots, e_4\}$ of $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{g}_3$ so that

$$\begin{aligned} &[e_2, e_3] = \lambda_1 e_1, && [e_3, e_1] = \lambda_2 e_2, \\ &[e_4, e_1] = \tfrac{1}{R}\{b e_1 - \lambda_2(\tfrac{a}{\lambda_1} + k_3) e_2\}, && [e_4, e_2] = \tfrac{1}{R}\{(a + k_3\lambda_1) e_1 + b e_2\}, \\ &[e_4, e_3] = \tfrac{1}{R}\{(c - k_2\lambda_1) e_1 + (d + k_1\lambda_2) e_2\}, \quad R > 0. && \end{aligned} \tag{8}$$

Lemma 3.1. *The group $\mathbb{R}e_4 \ltimes E(1,1)$ admits a non-symmetric Bach-flat left-invariant metric if and only if it is homothetic to a Lie group determined by a solvable Lie algebra given by Equation* (8) *with $\lambda_1 = 1$, $a = -k_3$, $c = k_2$, $d = -k_1\lambda_2$ and, in addition, one of the following holds:*

(i) $b = \pm R\sqrt{6 + 3\sqrt{3}}$*, and* $\lambda_2 = -2 - \sqrt{3}$.

(ii) $b = \pm R\sqrt{6 - 3\sqrt{3}}$*, and* $\lambda_2 = -2 + \sqrt{3}$.

Moreover, the Lie group $\mathbb{R}e_4 \ltimes E(2)$ does not admit any non-symmetric Bach-flat left-invariant metric.

Proof. We start analyzing the Bach tensor of $\mathbb{R}e_4 \ltimes E(1,1)$ and $\mathbb{R}e_4 \ltimes E(2)$. In order to simplify the expressions we use the notation $A = \frac{a}{\lambda_1} + k_3$, $C = c - k_2\lambda_1$ and $D = d + k_1\lambda_2$. Moreover, since the structure constants of $\mathfrak{g}_3$ satisfy $\lambda_1\lambda_2 \neq 0$, one may work with a homothetic basis $\widetilde{e}_k = \frac{1}{\lambda_1} e_k$ so that we may assume $\lambda_1 = 1$. A long but straightforward calculation shows that the components of the Bach tensor, with the structure constants in Equation (8), are given by

$$\begin{aligned} &\mathfrak{B}_{11} = \tfrac{1}{24R^4}\mathfrak{P}_{11},\ \mathfrak{B}_{12} = \tfrac{1}{12R^4}\mathfrak{P}_{12},\ \mathfrak{B}_{13} = \tfrac{1}{12R^4}\mathfrak{P}_{13},\ \mathfrak{B}_{14} = \tfrac{\lambda_2}{12R^3}\mathfrak{P}_{14}, \mathfrak{B}_{22} = \tfrac{1}{24R^4}\mathfrak{P}_{22}, \\ &\mathfrak{B}_{23} = \tfrac{1}{12R^4}\mathfrak{P}_{23},\ \mathfrak{B}_{24} = \tfrac{1}{12R^3}\mathfrak{P}_{24},\ \mathfrak{B}_{33} = \tfrac{1}{24R^4}\mathfrak{P}_{33},\ \mathfrak{B}_{34} = \tfrac{1}{12R^3}\mathfrak{P}_{34},\ \mathfrak{B}_{44} = \tfrac{1}{24R^4}\mathfrak{P}_{44}, \end{aligned} \tag{9}$$

where the polynomials $\mathfrak{P}_{ij}$'s correspond to:

$$\begin{aligned} \mathfrak{P}_{11} = {} & 12(A^2+R^2)^2\lambda_2^4 - 4(A^2+R^2)^2\lambda_2^3 - (20b^2 - C^2 - 8D^2)(A^2+R^2)\lambda_2^2 - 4(C^2+D^2)(5C^2+D^2) \\ & + (12A^4 - 4(2b^2 - 3C^2 - D^2 - 6R^2)A^2 - 42bCDA - 4R^2(2b^2 - 3C^2 - D^2 - 3R^2))\lambda_2 \\ & - 20A^4 + (28b^2 - 40C^2 + 3D^2 - 40R^2)A^2 - 42bCDA - 20R^4 + (3D^2 - 40C^2)R^2) + b^2(43C^2 + D^2 + 28R^2), \end{aligned}$$

$$\begin{aligned} \mathfrak{P}_{12} = {} & -16b(A^2+R^2)A\lambda_2^3 - 8CD(A^2+R^2)\lambda_2^2 - (5CDA^2 - b(5C^2 - 16D^2)A + 5CDR^2)\lambda_2 \\ & + 16bA^3 - 8CDA^2 + b(16C^2 - 5D^2 + 16R^2)A + CD(21b^2 - 8(C^2 + D^2 + R^2)), \end{aligned}$$

$$\begin{aligned} \mathfrak{P}_{13} = {} & -8AD(A^2+R^2)\lambda_2^3 + (4AD(A^2+R^2) - 3bCR^2)\lambda_2^2 + 3b(8CA^2 + 3bDA - 3b^2C + 8C(C^2+D^2+R^2)) \\ & + (DA^3 - 9bCA^2 + D(12b^2 + R^2 - 8(C^2+D^2))A - 12bCR^2)\lambda_2, \end{aligned}$$

$$\mathfrak{P}_{14} = -8D(A^2+R^2)\lambda_2^2 + (4DA^2 + 3bCA + 4DR^2)\lambda_2 + DA^2 + 3bCA + D(3b^2 + R^2 - 8(C^2+D^2)),$$

$$\begin{aligned} \mathfrak{P}_{22} = {} & -20(A^2+R^2)^2\lambda_2^4 + 12(A^2+R^2)^2\lambda_2^3 + (28b^2 + 3C^2 - 40D^2)(A^2+R^2)\lambda_2^2 + 12R^4 + (8C^2 + D^2)R^2 \\ & - (4A^4 + 4(2b^2 - C^2 - 3D^2 + 2R^2)A^2 - 42bCDA + 4R^2(2b^2 - C^2 - 3D^2 + R^2))\lambda_2 - 4(C^2+D^2)(C^2+5D^2) \\ & + 12A^4 - (20b^2 - 8C^2 - D^2 - 24R^2)A^2 + 42bCDA + b^2(C^2 + 43D^2 - 20R^2), \end{aligned}$$

$$\begin{aligned} \mathfrak{P}_{23} = {} & -(AC - 24bD)(A^2+R^2)\lambda_2^2 - (4(AC + 3bD)R^2 + A(4CA^2 + 9bDA + 9b^2C))\lambda_2 \\ & + 8CA^3 + 4C(2(C^2+D^2+R^2) - 3b^2)A - 3bD(3b^2 + R^2 - 8(C^2+D^2)), \end{aligned}$$

$$\mathfrak{P}_{24} = -C(A^2+R^2)\lambda_2^2 - (4CA^2 - 3bDA + 4CR^2)\lambda_2 + 8CA^2 + 3bDA - 3b^2C + 8C(C^2 + D^2 + R^2),$$

$$\mathfrak{P}_{33} = -4(A^2 - 3R^2)(A^2+R^2)\lambda_2^4 + 4(A^2 - 3R^2)(A^2+R^2)\lambda_2^3 - ((12b^2 + C^2 - 8D^2)A^2 + 3(4b^2 + C^2 - 8D^2)R^2)\lambda_2^2$$

$$
\begin{aligned}
&+2(2A^4+2(6b^2-C^2-D^2-2R^2)A^2+9bCDA+6R^2(2b^2-C^2-D^2-R^2))\lambda_2\\
&-4A^4-(12b^2-8C^2+D^2-8R^2)A^2-18bCDA+12R^4-3(4b^2-8C^2+D^2)R^2+3(C^2+D^2)(4(C^2+D^2)-19b^2),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{34} =& -8A(A^2+R^2)\lambda_2^4+8A(A^2+R^2)\lambda_2^3+A(C^2-8D^2)\lambda_2^2+(8A^3+4(C^2+D^2+2R^2)A+9bCD)\lambda_2\\
&-8A^3-9bCD-A(8C^2-D^2+8R^2),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{44} =& \,4(3A^2-R^2)(A^2+R^2)\lambda_2^4-4(3A^4+2R^2A^2-R^4)\lambda_2^3+((4b^2-3C^2+24D^2)A^2+(4b^2-C^2+8D^2)R^2)\lambda_2^2\\
&+(4R^4-4(2A^2+2b^2+C^2+D^2)R^2-2A(6A^3+4b^2A+6(C^2+D^2)A+9bCD))\lambda_2\\
&+12A^4+(4b^2+24C^2-3D^2+8R^2)A^2+18bCDA-4R^4+(4b^2+8C^2-D^2)R^2+(C^2+D^2)(13b^2+12(C^2+D^2)).
\end{aligned}
$$

Hence, $\mathbb{R}e_4 \ltimes E(1,1)$ or $\mathbb{R}e_4 \ltimes E(2)$ admits a Bach-flat left-invariant metric if and only if the structure constants in Equation (8) satisfy the equations $\{\mathfrak{P}_{ij}=0\}$. Let $\mathcal{I} \subset \mathbb{R}[A,b,\lambda_2,C,D,R]$ be the ideal generated by the polynomials $\mathfrak{P}_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the lexicographical order and a detailed analysis of that basis shows that the polynomial

$$
\begin{aligned}
\mathbf{g}_0 \;=\;& D^6(C^2+D^2)(2D^2+R^2)(25D^2+4R^2)(16D^2+5R^2)\\
\times\;& (9D^2+16R^2)(25D^2+24R^2)(80D^4+R^4-16D^2R^2)
\end{aligned}
\qquad (10)
$$

belongs to $\mathcal{G}$. Since the zero sets of $\{\mathfrak{P}_{ij}=0\}$ and $\mathcal{I}=\langle\mathfrak{P}_{ij}\rangle=\langle\mathcal{G}\rangle$ coincide, then necessarily $D=0$.

Next, we compute a Gröbner basis $\mathcal{G}_1$ of the ideal generated by $\mathcal{G}\cup\{D\}$ with respect to the lexicographical order and we get that the polynomial

$$
\mathbf{g}_1 = C^4(9C^2+4R^2)(25C^2+16R^2)(49C^2+24R^2)\lambda_2^3
$$

belongs to $\mathcal{G}_1$. Thus, since $\lambda_2 \neq 0$, we get $C=0$.

Now, for $C=D=0$, Equation (9) implies that

$$
\mathfrak{P}_{34} = -8(\lambda_2-1)^2A(A^2+R^2)(\lambda_2^2+\lambda_2+1)
$$

and therefore we are led to the following possibilities:

$$
(1)\,\lambda_2 = 1, \qquad (2)\,A=0.
$$

3.1. Case (1):

$C=0$, $D=0$, $\lambda_2=1$. In this case, a direct calculation shows that the corresponding Lie group given by Equation (8) is locally conformally flat and therefore a symmetric manifold [34].

3.2. Case (2):

$C=0$, $D=0$, $A=0$. Excluding $\lambda_2=1$ solved in the previous case, Equation (9) implies that the Bach-flat condition is equivalent to

$$
b^2-R^2(\lambda_2^2+\lambda_2+1)=0, \quad 3R^2-b^2(\lambda_2+4)=0,
$$

from where it easily follows that

$$
b=\pm R, \quad \lambda_2=-1,
$$

in which case a straightforward calculation shows that the manifold is Einstein and thus locally symmetric [24], or otherwise

$$
\begin{aligned}
b=\pm R\sqrt{6+3\sqrt{3}}, &\qquad \lambda_2=-2-\sqrt{3}, \quad \text{or}\\
b=\pm R\sqrt{6-3\sqrt{3}}, &\qquad \lambda_2=-2+\sqrt{3}\,.
\end{aligned}
$$

Note that $\lambda_1\lambda_2=\lambda_2<0$; hence the group is $\mathbb{R}e_4 \ltimes E(1,1)$ and a straightforward calculation shows that none of the cases above is locally symmetric. Thus, those cases correspond to Assertions (i) and (ii) in Lemma 3.1. This finishes the proof.

4. Left-invariant metrics on $\mathbb{R}e_4 \ltimes H^3$

Let $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{h}_3$ be a semi-direct extension of the Heisenberg algebra $\mathfrak{h}_3$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ and $\langle \cdot, \cdot \rangle_3$ its restriction to $\mathfrak{h}_3$. Then, there exists an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{h}_3$ such that (see [30])

$$[\mathbf{v}_3, \mathbf{v}_2] = 0, \qquad [\mathbf{v}_3, \mathbf{v}_1] = 0, \qquad [\mathbf{v}_1, \mathbf{v}_2] = \lambda_3 \mathbf{v}_3, \tag{11}$$

where $\lambda_3 \neq 0$ is a real number. The algebra of all derivations of $\mathfrak{h}_3$ is given with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ by

$$\operatorname{der}(\mathfrak{h}_3) = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ \hat{h} & \hat{f} & \alpha_{11} + \alpha_{22} \end{pmatrix} ; \alpha_{ij}, \hat{f}, \hat{h} \in \mathbb{R} \right\}.$$

We rotate the basis elements $\{\mathbf{v}_1, \mathbf{v}_2\}$ so that the matrix $A = (\alpha_{ij})$ decomposes as the sum of a diagonal matrix and a skew-symmetric matrix. Hence

$$\operatorname{der}(\mathfrak{h}_3) = \left\{ \begin{pmatrix} a & c & 0 \\ -c & d & 0 \\ h & f & a+d \end{pmatrix} ; a, c, d, f, h \in \mathbb{R} \right\},$$

and we consider the Lie algebra $\mathfrak{g} = \mathbb{R}\mathbf{v}_4 \oplus \mathfrak{h}_3$ given by

$$\begin{array}{lll} [\mathbf{v}_3, \mathbf{v}_2] = 0, & [\mathbf{v}_3, \mathbf{v}_1] = 0, & [\mathbf{v}_1, \mathbf{v}_2] = \gamma \mathbf{v}_3, \\ [\mathbf{v}_4, \mathbf{v}_1] = a\mathbf{v}_1 - c\mathbf{v}_2 + h\mathbf{v}_3, & [\mathbf{v}_4, \mathbf{v}_2] = c\mathbf{v}_1 + d\mathbf{v}_2 + f\mathbf{v}_3, & [\mathbf{v}_4, \mathbf{v}_3] = (a+d)\mathbf{v}_3. \end{array}$$

Since $\mathbb{R}\mathbf{v}_4$ needs not to be orthogonal to $\mathfrak{h}_3$, we set $k_i = \langle \mathbf{v}_i, \mathbf{v}_4 \rangle$, for $i = 1, 2, 3$. Let $\hat{e}_4 = \mathbf{v}_4 - \sum_i k_i \mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1, \ldots, e_4\}$ of $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{h}_3$ so that

$$\begin{array}{ll} [e_1, e_2] = \gamma e_3, & [e_4, e_1] = \frac{1}{R}\{a e_1 - c e_2 + (h + k_2\gamma) e_3\}, \\ [e_4, e_3] = \frac{1}{R}(a+d) e_3, & [e_4, e_2] = \frac{1}{R}\{c e_1 + d e_2 + (f - k_1\gamma) e_3\}, \quad R > 0. \end{array} \tag{12}$$

Lemma 4.1. *The group $\mathbb{R}e_4 \ltimes H^3$ admits a non-symmetric Bach-flat left-invariant metric if and only if it is homothetic to a Lie group determined by a solvable Lie algebra given by Equation (12) with $\gamma = 1$, $f = k_1$, $h = -k_2$ and, in addition, one of the following holds:*

(i) $a = d = \pm R$. In this case $\mathbb{R}e_4 \ltimes H^3$ is half conformally flat.

(ii) $c = 0$, $a = -\frac{R^2}{8d}$ and

(ii.a) $d = \pm\frac{1}{4} R\sqrt{7 - 3\sqrt{5}}$, or

(ii.b) $d = \pm\frac{1}{4} R\sqrt{7 + 3\sqrt{5}}$.

Proof. First we obtain the Bach tensor of $\mathbb{R}e_4 \ltimes H^3$. In order to simplify the expressions we use the notation $F = f - k_1\gamma$ and $H = h + k_2\gamma$. Moreover, since the structure constant of $\mathfrak{h}_3$ satisfies $\gamma \neq 0$, one may work with a homothetic basis $\widetilde{e}_k = \frac{1}{\gamma} e_k$ so that we may assume $\gamma = 1$. A long but straightforward calculation shows that the components of the Bach tensor, with the structure constants in Equation (12), are given by

$$\begin{array}{l} \mathfrak{B}_{11} = \frac{1}{24R^4}\mathfrak{P}_{11}, \; \mathfrak{B}_{12} = \frac{1}{12R^4}\mathfrak{P}_{12}, \; \mathfrak{B}_{13} = \frac{1}{12R^4}\mathfrak{P}_{13}, \; \mathfrak{B}_{14} = \frac{1}{12R^3}\mathfrak{P}_{14}, \mathfrak{B}_{22} = \frac{1}{24R^4}\mathfrak{P}_{22}, \\ \mathfrak{B}_{23} = \frac{1}{12R^4}\mathfrak{P}_{23}, \; \mathfrak{B}_{24} = \frac{1}{12R^3}\mathfrak{P}_{24}, \; \mathfrak{B}_{33} = \frac{1}{24R^4}\mathfrak{P}_{33}, \; \mathfrak{B}_{34} = 0, \qquad \mathfrak{B}_{44} = \frac{1}{24R^4}\mathfrak{P}_{44}, \end{array} \tag{13}$$

where the polynomials $\mathfrak{P}_{ij}$'s correspond to:

$$\begin{aligned} \mathfrak{P}_{11} = {} & 24ac^2d - 16a^3d + 48ad^3 + 84a^2c^2 + 16a^2d^2 - 108c^2d^2 + (F^2 - 20(H^2 + R^2))a^2 - 21(F^2 - H^2)c^2 \\ & - 3(4F^2 + 19H^2 + 4R^2)d^2 + 78FHac - 4(22H^2 + 7R^2)ad + 78FHcd - 4(F^2 + H^2 + R^2)(F^2 - 3(H^2 + R^2)), \end{aligned}$$

$$\mathfrak{P}_{12} = -58a^2cd + 58acd^2 - 18a^3c + 24ac^3 - 24c^3d + 18cd^3 - 12FHa^2 + 21FHc^2 - 12FHd^2$$

$$
\begin{aligned}
&+ (31F^2 - 2(4H^2 + R^2))ac - 53FHad + (8F^2 - 31H^2 + 2R^2)cd + 8FH(F^2 + H^2 + R^2),\\
\mathfrak{P}_{13} &= 53Facd - 3Fc^3 - 9Hd^3 + 33Fa^2c - 28Ha^2d + 3Hac^2 - 48Had^2 + 24Hc^2d - 9Fcd^2\\
&\quad + 16H(F^2 + H^2 + R^2)a - 8F(F^2 + H^2 + R^2)c + 24H(F^2 + H^2 + R^2)d,\\
\mathfrak{P}_{14} &= -3Fa^2 + 3Fc^2 + 3Hac - 14Fad - 15Hcd + 8F(F^2 + H^2 + R^2),\\
\mathfrak{P}_{22} &= 24ac^2d + 48a^3d - 16ad^3 - 108a^2c^2 + 16a^2d^2 + 84c^2d^2 - 3(19F^2 + 4(H^2 + R^2))a^2 + 21(F^2 - H^2)c^2\\
&\quad - (20F^2 - H^2 + 20R^2)d^2 - 78FHac - 4(22F^2 + 7R^2)ad - 78FHcd + 4(F^2 + H^2 + R^2)(3F^2 - H^2 + 3R^2),\\
\mathfrak{P}_{23} &= -53Hacd - 9Fa^3 + 3Hc^3 + 9Ha^2c - 48Fa^2d + 24Fac^2 - 28Fad^2 + 3Fc^2d - 33Hcd^2\\
&\quad + 24F(F^2 + H^2 + R^2)a + 8H(F^2 + H^2 + R^2)c + 16F(F^2 + H^2 + R^2)d,\\
\mathfrak{P}_{24} &= -3Hc^2 + 3Hd^2 - 15Fac + 14Had + 3Fcd - 8H(F^2 + H^2 + R^2),\\
\mathfrak{P}_{33} &= 24ac^2d - 16a^3d - 16ad^3 - 12a^2c^2 - 48a^2d^2 - 12c^2d^2 + (43F^2 + 28(H^2 + R^2))a^2 - 9(F^2 + H^2)c^2\\
&\quad + (28F^2 + 43H^2 + 28R^2)d^2 - 54FHac + (104(F^2 + H^2) + 44R^2)ad + 54FHcd - 20(F^2 + H^2 + R^2)^2,\\
\mathfrak{P}_{44} &= -72ac^2d - 16a^3d - 16ad^3 + 36a^2c^2 + 16a^2d^2 + 36c^2d^2 + (13F^2 + 4(H^2 + R^2))a^2 + 9(F^2 + H^2)c^2\\
&\quad + (4F^2 + 13H^2 + 4R^2)d^2 + 54FHac - (16(F^2 + H^2) - 12R^2)ad - 54FHcd + 4(3(F^2 + H^2) - R^2)(F^2 + H^2 + R^2).
\end{aligned}
$$

Therefore, $\mathbb{R}e_4 \ltimes H^3$ admits a Bach-flat left-invariant metric if and only if the structure constants given in Equation (12) satisfy the equations $\{\mathfrak{P}_{ij} = 0\}$. Let $\mathcal{I} \subset \mathbb{R}[a, c, d, H, F, R]$ be the ideal generated by the polynomials $\mathfrak{P}_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the lexicographical order and a detailed analysis of the Gröbner basis shows that the polynomial

$$
\begin{aligned}
\mathbf{g}_0 \;=\;& FR^4(2F^2 + R^2)^4(4F^2 + R^2)(F^2 + H^2 + R^2)^2(4F^2 + 9R^2)(9F^2 + 11R^2)\\
\times\;& ((F^2 - H^2)^2 + F^2R^2 + H^2R^2)(10000F^4 + 10200F^2R^2 + 3087R^4)\\
\times\;& (606208F^4 + 861952F^2R^2 + 144669R^4)
\end{aligned}
$$

belongs to $\mathcal{G}$. Since the zero sets of $\{\mathfrak{P}_{ij} = 0\}$ and $\mathcal{I} = \langle \mathfrak{P}_{ij} \rangle = \langle \mathcal{G} \rangle$ coincide and $R > 0$, then necessarily $F = 0$.

Next, we compute a Gröbner basis $\mathcal{G}'$ of the ideal generated by $\mathcal{G} \cup \{F\}$ with respect to the lexicographical order and we get that the polynomial

$$
\mathbf{g}_0' = H(H^2 + R^2)(4H^2 + R^2)(4H^2 + 9R^2)(9H^2 + 11R^2)
$$

belongs to $\mathcal{G}'$. Thus, we get $H = 0$.

Now, computing a Gröbner basis $\mathcal{G}''$ of the ideal generated by $\mathcal{G}' \cup \{H\}$ with respect to the graded reverse lexicographical order we find that the polynomial

$$
\mathbf{g}_0'' = (a - d)(24c^2 - 8ad - R^2)R^4
$$

belongs to $\mathcal{G}''$ and therefore we are led to the following possibilities:

$$
(1)\, a = d, \quad (2)\, 24c^2 - 8ad - R^2 = 0.
$$

4.1. Case (1):

$F = 0$, $H = 0$, $a = d$. In this case, Equation (13) implies that the Bach-flat condition is equivalent to

$$
4d^4 + R^4 - 5d^2R^2 = 0,
$$

from where we easily get

$$
d = \pm R \quad \text{or} \quad d = \pm \frac{R}{2}.
$$

If $d = \pm\frac{R}{2}$, the manifold is half conformally flat and Einstein, thus locally symmetric and homothetic to the complex hyperbolic plane. If $d = \pm R$, a direct calculation shows that the manifold is half conformally flat and non-symmetric, hence showing Assertion (i) in Lemma 4.1. Furthermore, it follows from [17] that all the Lie groups in Lemma 4.1(i) are isometric.

4.2. Case (2):

$F=0$, $H=0$, $24c^2-8ad-R^2=0$. Equation (13) implies that

$$\mathfrak{P}_{12} = -(a-d)c(18a^2+18d^2+68ad+R^2).$$

Since $a=d$ was already solved in the previous case, we compute a Gröbner basis $\mathcal{G}_2$ of the ideal generated by $\mathcal{G}'' \cup \{c(18a^2+18d^2+68ad+R^2)\} \subset \mathbb{R}[R,a,c,d,H,F]$ with respect to the lexicographical order and we get that the polynomial

$$\mathfrak{g}_2 = cd^4(25c^4+18c^2d^2+d^4)(961c^4+1298c^2d^2+121d^4)$$

belongs to $\mathcal{G}_2$. Thus, we have two possibilities:

$$(2.i)\; d=0, \qquad (2.ii)\; c=0.$$

4.2.1. Case (2.i):

$F=0$, $H=0$, $24c^2-8ad-R^2=0$, $d=0$. In this case, from Equation (13) we get that the Bach-flat condition is equivalent to

$$33a^2c^2-R^4=0, \quad ac(3a^2+4c^2)=0,$$

which does not hold since $R>0$.

4.2.2. Case (2.ii):

$F=0$, $H=0$, $24c^2-8ad-R^2=0$, $c=0$. Since $d=0$ was solved in the previous case, we have $a=-\frac{R^2}{8d}$ and Equation (13) implies that the Bach-flat condition is equivalent to

$$64d^4-56d^2R^2+R^4=0.$$

Thus, it follows that

$$d=\pm\frac{1}{4}R\sqrt{7-3\sqrt{5}} \quad \text{or} \quad d=\pm\frac{1}{4}R\sqrt{7+3\sqrt{5}},$$

and a straightforward calculation shows that none of these cases is locally symmetric. Therefore, they correspond to Assertions (ii.a) and (ii.b) in Lemma 4.1, finishing the proof.

5. Left-invariant metrics on $\mathbb{R}e_4 \ltimes \mathbb{R}^3$

Let $\mathfrak{g}=\mathbb{R}\ltimes\mathfrak{r}^3$ be a semi-direct extension of the Abelian Lie algebra $\mathfrak{r}^3$. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle_3$ its restriction to $\mathfrak{r}^3$. The algebra of all derivations $\mathfrak{D}$ of $\mathfrak{r}^3$ is $\mathfrak{gl}(3,\mathbb{R})$. If we fix $\mathfrak{D}\in\mathfrak{gl}(3,\mathbb{R})$, there exists a $\langle\cdot,\cdot\rangle_3$-orthonormal basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ of $\mathfrak{r}^3$ where $\mathfrak{D}$ decomposes as a sum of a diagonal matrix and a skew-symmetric matrix. Hence

$$\operatorname{der}(\mathfrak{r}^3) = \left\{ \begin{pmatrix} a & -b & -c \\ b & f & -h \\ c & h & p \end{pmatrix} ;\, a,b,c,f,h,p\in\mathbb{R} \right\}.$$

Now, the corresponding semi-direct product $\mathfrak{g}=\mathbb{R}\ltimes\mathfrak{r}^3$, is given by

$$\begin{array}{lll} [\mathbf{v}_1,\mathbf{v}_2]=0, & [\mathbf{v}_1,\mathbf{v}_3]=0, & [\mathbf{v}_2,\mathbf{v}_3]=0, \\ [\mathbf{v}_4,\mathbf{v}_1]=a\mathbf{v}_1+b\mathbf{v}_2+c\mathbf{v}_3, & [\mathbf{v}_4,\mathbf{v}_2]=-b\mathbf{v}_1+f\mathbf{v}_2+h\mathbf{v}_3, & [\mathbf{v}_4,\mathbf{v}_3]=-c\mathbf{v}_1-h\mathbf{v}_2+p\mathbf{v}_3, \end{array}$$

with respect to some basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$ so that $\mathfrak{g}=\mathbb{R}\mathbf{v}_4\oplus\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$. Since $\mathbb{R}\mathbf{v}_4$ needs not to be orthogonal to $\mathfrak{r}^3$, set $k_i=\langle\mathbf{v}_i,\mathbf{v}_4\rangle$, for $i=1,2,3$. Let $\hat{e}_4=\mathbf{v}_4-\sum_i k_i\mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1,\dots,e_4\}$ of $\mathfrak{g}=\mathbb{R}\oplus\mathfrak{r}^3$ so that

$$\begin{aligned} &[e_4,e_1]=\tfrac{1}{R}\{ae_1+be_2+ce_3\}, \qquad [e_4,e_2]=\tfrac{1}{R}\{-be_1+fe_2+he_3\}, \\ &[e_4,e_3]=\tfrac{1}{R}\{-ce_1-he_2+pe_3\}, \quad R>0. \end{aligned} \tag{14}$$

Lemma 5.1. *The group $\mathbb{R}e_4 \ltimes \mathbb{R}^3$ admits a non-symmetric Bach-flat left-invariant metric if and only if it is homothetic to a Lie group determined by a solvable Lie algebra given by Equation* (14) *with $a = 1$ and, in addition, one of the following holds:*

(i) $f = p = \frac{1}{4}$, *and* $b = c = 0$.
(ii) $f \neq p$, $h = 0$ *and*
  (ii.a) $b = c = 0$, *and* $f = (1 + \sqrt{p})^2$, *with* $p > 0$.
  (ii.b) $b = c = 0$, *and* $f = (-1 + \sqrt{p})^2$, *with* $p > 0$, $p \neq 1$.
  (ii.c) $b = 0$, $f = 4$ *and* $p = 1$.
  (ii.d) $c = 0$, $f = 1$, $p = 4$ *and* $b \neq 0$.

Proof. A long but straightforward calculation shows that the components of the Bach tensor of $\mathbb{R}e_4 \ltimes \mathbb{R}^3$, with the structure constants in Equation (14), are given by

$$\begin{aligned} &\mathfrak{B}_{11} = \tfrac{1}{6R^4}\mathfrak{P}_{11},\ \mathfrak{B}_{12} = \tfrac{1}{6R^4}\mathfrak{P}_{12},\ \mathfrak{B}_{13} = \tfrac{1}{6R^4}\mathfrak{P}_{13},\ \mathfrak{B}_{14} = 0,\ \mathfrak{B}_{22} = \tfrac{1}{6R^4}\mathfrak{P}_{22},\\ &\mathfrak{B}_{23} = \tfrac{1}{6R^4}\mathfrak{P}_{23},\ \mathfrak{B}_{24} = 0, \qquad \mathfrak{B}_{33} = \tfrac{1}{6R^4}\mathfrak{P}_{33},\ \mathfrak{B}_{34} = 0,\ \mathfrak{B}_{44} = \tfrac{1}{6R^4}\mathfrak{P}_{44}, \end{aligned} \tag{15}$$

where the polynomials $\mathfrak{P}_{ij}$'s correspond to:

$$\begin{aligned} \mathfrak{P}_{11} &= a^4 + 9a^2b^2 + 9a^2c^2 - (f+p)a^3 + 6(f+2p)ab^2 + 6(2f+p)ac^2 - (2f^2+2p^2+7fp)a^2\\ &\quad - 3f(5f+4p)b^2 - 3p(4f+5p)c^2 + 18h(f-p)bc + 3(f+p)(f^2+p^2)a - (f-p)^2(f^2+3h^2+p^2+fp),\\ \mathfrak{P}_{12} &= -12abc^2 - 2a^3b - 12ab^3 + 12fb^3 + 2(9f+5p)a^2b + 6ha^2c + 3(f+3p)bc^2\\ &\quad - (18f^2+3h^2-p^2)ab + 6h(2f-p)ac + (2f^3+12fh^2-10f^2p-fp^2-9h^2p)b + 6h(f+p)(f-2p)c,\\ \mathfrak{P}_{13} &= -12ab^2c - 2a^3c - 12ac^3 + 12pc^3 - 6ha^2b + 2(5f+9p)a^2c + 3(3f+p)b^2c\\ &\quad + 6h(f-2p)ab + (f^2-3h^2-18p^2)ac + 6h(f+p)(2f-p)b + (2p^3-9fh^2-f^2p-10fp^2+12h^2p)c,\\ \mathfrak{P}_{22} &= -a^4 - 15a^2b^2 - 3a^2c^2 - 18habc + (3f+p)a^3 + 6(f-2p)ab^2 + 6pac^2 - f(2f-3p)a^2 + 3f(3f+4p)b^2\\ &\quad - 3p^2c^2 + 18hpbc - (f^3-p^3-12fh^2+7f^2p-3fp^2+12h^2p)a + (f-p)(f^3+p^3+9fh^2-2fp^2+15h^2p),\\ \mathfrak{P}_{23} &= -12a^2bc - 6(f+p)abc + 9hab^2 - 9hac^2 + h(f-p)a^2 - 3h(4f-p)b^2\\ &\quad - 3h(f-4p)c^2 + 6(f+p)^2bc + 10h(f^2-p^2)a - 2h(f-p)(f^2-8pf+6h^2+p^2),\\ \mathfrak{P}_{33} &= -a^4 - 3a^2b^2 - 15a^2c^2 + 18habc + (f+3p)a^3 + 6fab^2 - 6(2f-p)ac^2 + p(3f-2p)a^2 - 3f^2b^2 + 3p(4f+3p)c^2\\ &\quad - 18fhbc + (f^3-p^3+3f^2p-7fp^2-12fh^2+12h^2p)a - (f-p)(f^3+p^3-2f^2p+15fh^2+9h^2p),\\ \mathfrak{P}_{44} &= a^4 + 9a^2b^2 + 9a^2c^2 - 3(f+p)a^3 - 18fab^2 - 18pac^2 + (4f^2+4p^2+fp)a^2\\ &\quad + 9f^2b^2 + 9p^2c^2 - (f+p)(3f^2+3p^2-4fp)a + (f-p)^2(f^2+9h^2+p^2-fp). \end{aligned}$$

Hence, $\mathbb{R}e_4 \ltimes \mathbb{R}^3$ admits a Bach-flat left-invariant metric if and only if the structure constants in Equation (14) satisfy the equations $\{\mathfrak{P}_{ij} = 0\}$. We consider separately the cases $a = 0$ and $a \neq 0$.

5.1. Case $a = 0$

Let $\mathcal{I}_0 \subset \mathbb{R}[b, f, c, h, p]$ be the ideal generated by the seven polynomials $\mathfrak{P}_{ij}$ in Equation (15). We compute a Gröbner basis $\mathcal{G}_0$ of $\mathcal{I}_0$ with respect to the graded reverse lexicographical order and get that it contains the polynomial

$$\mathbf{g}_0 = p^8(f-p)^2.$$

Since the zero sets of $\{\mathfrak{P}_{ij} = 0\}$ and $\mathcal{I}_0 = \langle\mathfrak{P}_{ij}\rangle = \langle\mathcal{G}_0\rangle$ coincide, we are led to the following cases:

$$(1)\, p = 0, \qquad (2)\, f = p.$$

5.1.1. Case (1):

$a = 0$, $p = 0$. In this case, one checks using Equation (15) that

$$\mathfrak{P}_{44} = f^2(9b^2 + f^2 + 9h^2)$$

and therefore necessarily $f = 0$. Now, a direct calculation shows that, in such a case, the manifold is Einstein and therefore symmetric [24].

5.1.2. Case (2):
$a=0$, $f=p$. Equation (15) implies that

$$\mathfrak{P}_{44}=9(b^2+c^2)p^2.$$

Since $p=0$ corresponds to Case (1) (§5.1.1), we have $b=c=0$ and a direct calculation shows that the manifold is locally conformally flat and thus symmetric [34].

5.2. Case $a\neq 0$
Taking $a\neq 0$ in Equation (14), we may work with a homothetic basis $\widetilde{e}_k=\frac{1}{a}e_k$ so that we may assume, without loss of generality, $a=1$.

Let $\mathcal{I}\subset\mathbb{R}[p,f,b,c,h]$ be the ideal generated by the seven polynomials $\mathfrak{P}_{ij}$ in Equation (15). Computing a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the lexicographical order we find that the following polynomial is in the basis:

$$\begin{aligned}\mathbf{g}=&(f-1)ch^2(24h^6+69h^4-12h^2-8)(128h^6+63h^4-324h^2-216)(16h^2+9)\\&\times(16h^2+81)(80h^4+95h^2+32)(464h^4+2175h^2+1824)(2116h^4+4884h^2+1089)\\&\times(49532953600h^{12}+100931329200h^{10}+67210421265h^8+16039857600h^6\\&\quad+1904414976h^4+177168384h^2+11943936)\\&\times(14705175456768h^{12}-11441136851376h^{10}+3165906982755h^8+580502490560h^6\\&\quad+263837594880h^4+2944180224h^2+127844352).\end{aligned}$$

Note that only the first five factors provide real roots, so we consider the following cases:

(1) $f=1$, (2) $c=0$, (3) $h=0$, (4) $(24h^6+69h^4-12h^2-8)(128h^6+63h^4-324h^2-216)=0$.

5.2.1. Case (1):
$a=1$, $f=1$. We compute a Gröbner basis $\mathcal{G}_1$ of the ideal generated by $\mathcal{G}\cup\{f-1\}\subset\mathbb{R}[p,f,b,c,h]$ with respect to the lexicographical order, and we get that

$$\mathbf{g}_1=(p-1)c^2\quad\text{and}\quad\mathbf{g}_1'=(p-1)h^2$$

belong to $\mathcal{G}_1$. Thus, we have two possibilities:

$$(1.i)\; p=1,\qquad (1.ii)\; c=h=0.$$

<u>*Case (1.i)*</u>: $a=1$, $f=1$, $p=1$. In this case, a direct calculation shows that the manifold is Einstein and therefore symmetric [24].
<u>*Case (1.ii)*</u>: $a=1$, $f=1$, $c=h=0$. Equation (15) implies that

$$\mathfrak{P}_{44}=(p-1)^2p(p-4).$$

Note that $p=1$ corresponds to the previous case and for $p=0$ a direct calculation shows that the manifold is locally conformally flat and thus symmetric [34].

Now, if $p=4$, Equation (15) shows that the manifold is Bach-flat and, moreover, one easily checks that it is non-symmetric. This is a particular case of Assertion (ii.b) in Lemma 5.1 if $b=0$ and it corresponds to Assertion (ii.d) if $b\neq 0$.

5.2.2. Case (2):
$a=1$, $c=0$. We consider the ideal generated by $\mathcal{G}\cup\{c\}\subset\mathbb{R}[p,h,f,b,c]$ and compute a Gröbner basis $\mathcal{G}_2$ for it with respect to the lexicographical order, obtaining that the polynomial

$$\begin{aligned}\mathbf{g}_2\;=\;&(f-1)b^2(b^4+90b^2+81)(5b^4+5b^2+2)(25b^4+2b^2+1)\\\times\;&(49b^4+138b^2+9)(725b^4+8613b^2+2850)(2116b^4+4884b^2+1089)\end{aligned}$$

belongs to $\mathcal{G}_2$. Excluding $f = 1$ solved in Case (1) (§5.2.1), the only real root for $\mathbf{g}_2$ corresponds to the factor b^2, so necessarily $b = 0$.

Next, we compute a new Gröbner basis $\mathcal{G}'_2$ for the ideal generated by $\mathcal{G}_2 \cup \{b\} \subset \mathbb{R}[p, f, b, c, h]$ with respect to the lexicographical order and we find that the polynomials

$$\mathbf{g}'_2 = (f-1)(4f-1)h^2(8h^2-1)(8h^2+3)(8h^2+9),$$
$$\mathbf{g}''_2 = (f-1)(4f-1)h^2(320fh^4+128h^4+320fh^2+40f^2+152h^2-5f+4)$$

belong to $\mathcal{G}'_2$. As a consequence, and since $f = 1$ was solved in Case (1) (§5.2.1), one easily checks that we have two possibilities:

$$(2.i)\, f = \tfrac{1}{4}, \qquad (2.ii)\, h = 0.$$

Case (2.i): $a = 1$, $c = 0$, $b = 0$, $f = \frac{1}{4}$. Computing a Gröbner basis $\mathcal{G}_{21}$ for the ideal generated by $\mathcal{G}'_2 \cup \{4f-1\} \subset \mathbb{R}[p, f, b, c, h]$ with respect to the lexicographical order we get that the polynomial

$$\mathbf{g}_{21} = (4p-1)(4p-9)$$

belongs to $\mathcal{G}_{21}$. Now, we have:

- If $p = \frac{1}{4}$, Equation (15) implies that the manifold is Bach-flat and, moreover, one easily checks that it is non-symmetric, corresponding to Assertion (i) in Lemma 5.1.
- If $p = \frac{9}{4}$, then we use again Equation (15) to get that the Bach-flat condition is equivalent to $h = 0$ and, in such a case, a direct calculation shows that the manifold is non-symmetric. This is a particular case of Assertion (ii.b) in Lemma 5.1.

Case (2.ii): $a = 1$, $c = 0$, $b = 0$, $h = 0$. We compute a Gröbner basis $\mathcal{G}_{22}$ for the ideal generated by $\mathcal{G}'_2 \cup \{h\} \subset \mathbb{R}[p, f, b, c, h]$ with respect to the lexicographical order and we find that the polynomial

$$\mathbf{g}_{22} = (f-1)((f-p)^2-2f-2p+1)(f^2+f+1)$$

belongs to $\mathcal{G}_{22}$. Excluding $f = 1$ solved in Case (1) (§5.2.1), it follows that necessarily $f = (\pm 1 + \sqrt{p})^2$ and Equation (15) shows that the manifold is Bach-flat. Moreover, a straightforward calculation shows that if $f = 0$ or $p = 0$ the manifold is locally conformally flat and thus symmetric [34], while it is non-symmetric if $f \cdot p \neq 0$. This last case corresponds to Assertions (ii.a) and (ii.b) in Lemma 5.1.

5.2.3. Case (3):

$a = 1$, $h = 0$. We consider the ideal generated by $\mathcal{G} \cup \{h\} \subset \mathbb{R}[p, f, b, c, h]$ and compute a Gröbner basis $\mathcal{G}_3$ for it with respect to the lexicographical order, obtaining that the polynomial

$$\mathbf{g}_3 = (f-1)cb(14c^2+33)(5c^4-25c^2+32)(1421c^4+28623c^2+45600)$$

belongs to $\mathcal{G}_3$. Since $f = 1$ and $c = 0$ were solved in the previous cases, we get that necessarily $b = 0$.

Next, we compute a Gröbner basis $\mathcal{G}'_3$ for the ideal generated by $\mathcal{G}_3 \cup \{b\} \subset \mathbb{R}[p, f, b, c, h]$ with respect to the lexicographical order and the polynomial

$$\mathbf{g}'_3 = (f-1)c^2(f-4)f(c^4+90c^2+81)(25c^4+2c^2+1)(49c^4+138c^2+9)$$

belongs to $\mathcal{G}'_3$. As a consequence, we must consider the following two possibilities:

$$(3.i)\, f = 0, \qquad (3.ii)\, f = 4.$$

Case (3.i): $a = 1$, $h = 0$, $b = 0$, $f = 0$. Equation (15) implies that the Bach-flat condition is equivalent to $p = 1$ and, in that case, a direct calculation shows that the manifold is locally conformally flat, and thus symmetric [34].

Case (3.ii): $a = 1$, $h = 0$, $b = 0$, $f = 4$. Assuming $c \neq 0$, since it was solved in Case (2) (§5.2.2), a straightforward calculation using Equation (15) shows that the Bach-flat condition is equivalent to $p = 1$. Moreover, a direct calculation shows that, in such a case, the manifold is not symmetric, corresponding to Assertion (ii.c) in Lemma 5.1.

5.2.4. Case (4):

$a = 1$, $(24h^6 + 69h^4 - 12h^2 - 8)(128h^6 + 63h^4 - 324h^2 - 216) = 0$. In this last case, it is hard to get a good Gröbner basis if we use $\mathcal{G}$ as the starting point as in the previous cases. Instead, we analyze in detail the polynomials in $\mathcal{G}$ (39 specifically) and we find that excluding the factors previously solved (i.e., factors involving $f - 1$, c and h), just one of those polynomials depends only on c and h and has the form

$$\mathfrak{g}_4 = (f-1)ch^2 Q(c,h)$$

where $Q(c,h) = \delta c^4 + S(h)c^2 + T(h)$, with $\delta > 0$ and where $S(h)$, $T(h)$ are polynomials with only even powers of h.

In the last step, we use the polynomial $Q(c,h)$ to compute a Gröbner basis $\mathcal{G}_4$ of the ideal generated by

$$Q(c,h) \cup \{(24h^6 + 69h^4 - 12h^2 - 8)(128h^6 + 63h^4 - 324h^2 - 216)\} \subset \mathbb{R}[c,h]$$

with respect to the graded reverse lexicographical order and we find that

$$\begin{aligned} \mathfrak{g}_4' \;&=\; 9408954328h^8 + 3490462417c^4h^4 + 8504049964c^2h^6 + 631105440c^6 \\ &+\; 48352913472h^6 + 4976629248c^4h^2 + 38523345312c^2h^4 + 5583229368c^4 \\ &+\; 72029134968h^4 + 37011199020c^2h^2 + 10563992784c^2 + 38487215664h^2 \\ &+\; 5890415904 \end{aligned}$$

belongs to $\mathcal{G}_4$. Therefore we conclude that there is no solution in this case, finishing the proof.

6. Left-invariant metrics on $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$

Let $\mathfrak{g} = \mathfrak{g}_3 \times \mathbb{R}$ be a direct extension of the unimodular Lie algebra $\mathfrak{g}_3 = \mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{g}_3 = \mathfrak{su}(2)$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ and let $\langle \cdot, \cdot \rangle_3$ denote its restriction to $\mathfrak{g}_3$. Following the work of Milnor [30], there exists an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{g}_3$ such that

$$[\mathbf{v}_2, \mathbf{v}_3] = \lambda_1 \mathbf{v}_1, \qquad [\mathbf{v}_3, \mathbf{v}_1] = \lambda_2 \mathbf{v}_2, \qquad [\mathbf{v}_1, \mathbf{v}_2] = \lambda_3 \mathbf{v}_3, \tag{16}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\lambda_1\lambda_2\lambda_3 \neq 0$. Moreover, the associated Lie group corresponds to $SU(2)$ (resp., $SL(2,\mathbb{R})$) if $\lambda_1, \lambda_2, \lambda_3$ are all positive (resp., if any of $\lambda_1, \lambda_2, \lambda_3$ is negative).

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis of $\mathfrak{g}$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are given by Equation (16) and $\mathfrak{g} = \mathfrak{g}_3 \oplus \mathbb{R}\mathbf{v}_4$. Since $\mathbb{R}\mathbf{v}_4$ needs not to be orthogonal to $\mathfrak{g}_3$, set $k_i = \langle \mathbf{v}_i, \mathbf{v}_4 \rangle$, for $i = 1, 2, 3$. Let $\hat{e}_4 = \mathbf{v}_4 - \sum_i k_i \mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1, \dots, e_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \oplus \mathbb{R}$ so that

$$\begin{aligned} &[e_1, e_2] = \lambda_3 e_3, \qquad [e_2, e_3] = \lambda_1 e_1, \qquad [e_3, e_1] = \lambda_2 e_2, \\ &[e_1, e_4] = \tfrac{1}{R}(k_3\lambda_2 e_2 - k_2\lambda_3 e_3), \qquad [e_2, e_4] = \tfrac{1}{R}(k_1\lambda_3 e_3 - k_3\lambda_1 e_1), \\ &[e_3, e_4] = \tfrac{1}{R}(k_2\lambda_1 e_1 - k_1\lambda_2 e_2), \qquad R > 0. \end{aligned} \tag{17}$$

Lemma 6.1. *The Lie groups $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$ do not admit any non-symmetric Bach-flat left-invariant metric.*

Proof. Since the structure constants of $\mathfrak{g}_3$ satisfy $\lambda_1\lambda_2\lambda_3 \neq 0$, one may work with a homothetic basis $\widetilde{e}_k = \frac{1}{\lambda_1} e_k$ so that we may assume $\lambda_1 = 1$. A long but straightforward calculation shows that the components of the Bach tensor of $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$ or $SU(2) \times \mathbb{R}$, with the structure constants in Equation (17), are given by

$$\begin{aligned} &\mathfrak{B}_{11} = \tfrac{1}{24R^4}\mathfrak{P}_{11},\ \mathfrak{B}_{12} = \tfrac{1}{12R^4}\mathfrak{P}_{12},\ \mathfrak{B}_{13} = \tfrac{1}{12R^4}\mathfrak{P}_{13},\ \mathfrak{B}_{14} = \tfrac{1}{12R^3}\mathfrak{P}_{14},\ \mathfrak{B}_{22} = \tfrac{1}{24R^4}\mathfrak{P}_{22}, \\ &\mathfrak{B}_{23} = \tfrac{1}{12R^4}\mathfrak{P}_{23},\ \mathfrak{B}_{24} = \tfrac{1}{12R^3}\mathfrak{P}_{24},\ \mathfrak{B}_{33} = \tfrac{1}{24R^4}\mathfrak{P}_{33},\ \mathfrak{B}_{34} = \tfrac{1}{12R^3}\mathfrak{P}_{34},\ \mathfrak{B}_{44} = \tfrac{1}{24R^4}\mathfrak{P}_{44}, \end{aligned} \tag{18}$$

where the polynomials $\mathfrak{P}_{ij}$'s correspond to:

$$
\begin{aligned}
\mathfrak{P}_{11} = {}& -4(\lambda_2-\lambda_3)^2(\lambda_2^2+\lambda_3^2+\lambda_2\lambda_3)k_1^4+4(3\lambda_3^4-\lambda_3^3+3\lambda_3-5)k_2^4+4(3\lambda_2^4-\lambda_2^3+3\lambda_2-5)k_3^4\\
&-(((\lambda_3-4)\lambda_3+24)\lambda_2^2-(8\lambda_3^2+4\lambda_3+3)\lambda_3^2+2(2\lambda_3^2+\lambda_3-6)\lambda_2\lambda_3)k_1^2k_2^2\\
&+(8\lambda_2^4-4(\lambda_3-1)\lambda_2^3-(\lambda_3-1)(\lambda_3+3)\lambda_2^2-24\lambda_3^2+4(\lambda_3+3)\lambda_2\lambda_3)k_1^2k_3^2\\
&+((4(6\lambda_3-1)\lambda_3+1)\lambda_2^2-2(2\lambda_3^2+\lambda_3-6)\lambda_2+(\lambda_3+12)\lambda_3-40)k_2^2k_3^2\\
&+R^2(\lambda_2-\lambda_3)^2(8\lambda_2^2+8\lambda_3^2+(8\lambda_3+4)\lambda_2+4\lambda_3+3)k_1^2\\
&-R^2(\lambda_3-1)((3\lambda_3+1)\lambda_2^2+4((3\lambda_3+2)\lambda_3+3)\lambda_2-8(((3\lambda_3+2)\lambda_3+2)\lambda_3+5))k_2^2\\
&-R^2(\lambda_2-1)((3\lambda_2+1)\lambda_3^2+4((3\lambda_2+2)\lambda_2+3)\lambda_3-8(((3\lambda_2+2)\lambda_2+2)\lambda_2+5))k_3^2\\
&+4R^4(3\lambda_2^4-(3\lambda_3+1)\lambda_2^3+(3\lambda_3-1)\lambda_3^3+\lambda_2^2\lambda_3+((-3\lambda_3^2+\lambda_3-1)\lambda_3+3)\lambda_2+3\lambda_3-5),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{12} = {}& (\lambda_2-\lambda_3^2)((8\lambda_2+5)\lambda_2+8)k_1k_2k_3^2-(8\lambda_3^4-8\lambda_2^3-(\lambda_3-4)\lambda_2^2\lambda_3-(4\lambda_3-1)\lambda_2\lambda_3^2)k_1^3k_2\\
&-(8\lambda_3^4-4\lambda_3^3-\lambda_3^2+(\lambda_3+4)\lambda_2\lambda_3-8\lambda_2)k_1k_2^3\\
&-R^2(8\lambda_3^4-4(\lambda_2+1)\lambda_3^3-((\lambda_2-3)\lambda_2+1)\lambda_3^2+10(\lambda_2+1)\lambda_2\lambda_3-((8\lambda_2+5)\lambda_2+8)\lambda_2)k_1k_2,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{13} = {}& -(\lambda_2^2-\lambda_3)((8\lambda_3+5)\lambda_3+8)k_1k_2^2k_3-(8\lambda_2^4-8\lambda_3^3-(4\lambda_2-1)\lambda_2^2\lambda_3-(\lambda_2-4)\lambda_2\lambda_3^2)k_1^3k_3\\
&+(8\lambda_3-(8\lambda_2^3-4\lambda_2^2+\lambda_2\lambda_3-\lambda_2+4\lambda_3)\lambda_2)k_1k_3^3\\
&-R^2(8\lambda_2^4-4(\lambda_3+1)\lambda_2^3-((\lambda_3-3)\lambda_3+1)\lambda_2^2+10(\lambda_3+1)\lambda_2\lambda_3-((8\lambda_3+5)\lambda_3+8)\lambda_3)k_1k_3,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{14} = {}& -8(\lambda_2-\lambda_3)^2(\lambda_2^2+\lambda_3^2+\lambda_3\lambda_2)k_1^3-(8\lambda_3^4-4(\lambda_2+1)\lambda_3^3-(\lambda_2-1)^2\lambda_3^2+8\lambda_2^2-4(\lambda_2+1)\lambda_2\lambda_3)k_1k_2^2\\
&-(8\lambda_2^4-4(\lambda_3+1)\lambda_2^3-(\lambda_3-1)^2\lambda_2^2+8\lambda_3^2-4(\lambda_3+1)\lambda_2\lambda_3)k_1k_3^2\\
&-R^2(\lambda_2-\lambda_3)^2(8\lambda_2^2+8\lambda_3^2+(8\lambda_3-4)\lambda_2-4\lambda_3-1)k_1,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{22} = {}& -4(5\lambda_2^4-3\lambda_3^4-3\lambda_2^3\lambda_3+\lambda_2\lambda_3^3)k_1^4-4(\lambda_3^4-\lambda_3^3-\lambda_3+1)k_2^4-4(5\lambda_2^4-3\lambda_2^3+\lambda_2-3)k_3^4\\
&+(3((\lambda_3+4)\lambda_3-8)\lambda_2^2+(8\lambda_3^2-4\lambda_3-1)\lambda_3^2+2((2\lambda_3-1)\lambda_3+2)\lambda_2\lambda_3)k_1^2k_2^2\\
&-(40\lambda_2^4-12(\lambda_3+1)\lambda_2^3-(\lambda_3-1)^2\lambda_2^2-24\lambda_3^2+4(\lambda_3+1)\lambda_2\lambda_3)k_1^2k_3^2\\
&-(3(8\lambda_3^2-4\lambda_3-1)\lambda_2^2-(4\lambda_3^2-2\lambda_3+4)\lambda_2+(\lambda_3+4)\lambda_3-8)k_2^2k_3^2\\
&-R^2(40\lambda_2^4-12(2\lambda_3+1)\lambda_2^3+(4\lambda_3-1)\lambda_2^2-3(8\lambda_3^2-4\lambda_3-1)\lambda_3^2-2(-4\lambda_3^2+2\lambda_3+1)\lambda_2\lambda_3)k_1^2\\
&+R^2(\lambda_3-1)^2(3\lambda_2^2+4(\lambda_3+1)\lambda_2+8(\lambda_3^2+\lambda_3+1))k_2^2\\
&+R^2(\lambda_2-1)((\lambda_2+3)\lambda_3^2+4((3\lambda_2+2)\lambda_2+3)\lambda_3-8((\lambda_2(5\lambda_2+2)+2)\lambda_2+3))k_3^2\\
&-4R^4(5\lambda_2^4-3(\lambda_3+1)\lambda_2^3+\lambda_2^2\lambda_3+(\lambda_3-1)^2(\lambda_3+1)\lambda_2-3(\lambda_3^4-\lambda_3^3-\lambda_3+1)),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{23} = {}& (\lambda_2\lambda_3-1)(8\lambda_2^2+8\lambda_3^2+5\lambda_2\lambda_3)k_1^2k_2k_3+((\lambda_3+(8\lambda_3^2-4\lambda_3-1)\lambda_2+4)\lambda_3-8)k_2^3k_3\\
&+(((8\lambda_2-4)\lambda_2\lambda_3+\lambda_2-\lambda_3+4)\lambda_2-8)k_2k_3^3\\
&+R^2(8\lambda_2^3\lambda_3+(5(\lambda_3-2)\lambda_3+1)\lambda_2^2+((2\lambda_3-3)(4\lambda_3+1)\lambda_3+4)\lambda_2+(\lambda_3+4)\lambda_3-8)k_2k_3,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{24} = {}& -8(\lambda_3^4-\lambda_3^3-\lambda_3+1)k_2^3-(8\lambda_3^4-4(\lambda_2+1)\lambda_3^3+8\lambda_2^2-(\lambda_2-1)^2\lambda_3^2-4(\lambda_2+1)\lambda_2\lambda_3)k_1^2k_2\\
&-((8\lambda_3^2-4\lambda_3-1)\lambda_2^2-(4\lambda_3^2-2\lambda_3+4)\lambda_2-(\lambda_3+4)\lambda_3+8)k_2k_3^2\\
&+R^2(\lambda_3-1)^2(\lambda_2^2+4(\lambda_3+1)\lambda_2-8(\lambda_3^2+\lambda_3+1))k_2,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{33} = {}& 4(3\lambda_2^4-5\lambda_3^4-\lambda_2^3\lambda_3+3\lambda_2\lambda_3^3)k_1^4-4(5\lambda_3^4-3\lambda_3^3+\lambda_3-3)k_2^4-4(\lambda_2^4-\lambda_2^3-\lambda_2+1)k_3^4\\
&-(40\lambda_3^4-12(\lambda_2+1)\lambda_3^3-24\lambda_2^2-(\lambda_2-1)^2\lambda_3^2+4(\lambda_2+1)\lambda_2\lambda_3)k_1^2k_2^2\\
&+(8\lambda_2^4+4(\lambda_3-1)\lambda_2^3+(\lambda_3-1)(3\lambda_3+1)\lambda_2^2-24\lambda_3^2+4(3\lambda_3+1)\lambda_2\lambda_3)k_1^2k_3^2\\
&-((4(6\lambda_3-1)\lambda_3+1)\lambda_2^2-2(2\lambda_3+1)(3\lambda_3-2)\lambda_2-(3\lambda_3+4)\lambda_3-8)k_2^2k_3^2\\
&+R^2(24\lambda_2^4-40\lambda_3^4-4(2\lambda_3+3)\lambda_2^3+12\lambda_3^3+(4\lambda_3-3)\lambda_2^2+\lambda_3^2+2(2(6\lambda_3-1)\lambda_3+1)\lambda_2\lambda_3)k_1^2\\
&+R^2(\lambda_3-1)((\lambda_3+3)\lambda_2^2+4((3\lambda_3+2)\lambda_3+3)\lambda_2-8(((5\lambda_3+2)\lambda_3+2)\lambda_3+3))k_2^2\\
&+R^2(\lambda_2-1)^2(8\lambda_2^2+4(\lambda_3+2)\lambda_2+(3\lambda_3+4)\lambda_3+8)k_3^2\\
&-4R^4(5\lambda_3^4-3(\lambda_2+1)\lambda_3^3+\lambda_2\lambda_3^2+(\lambda_2-1)^2(\lambda_2+1)\lambda_3-3(\lambda_2-1)^2(\lambda_2^2+\lambda_2+1)),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{34} = {}& -8(\lambda_2^4-\lambda_2^3-\lambda_2+1)k_3^3-(8\lambda_2^4-4(\lambda_3+1)\lambda_2^3-(\lambda_3-1)^2\lambda_2^2+8\lambda_3^2-4(\lambda_3+1)\lambda_2\lambda_3)k_1^2k_3\\
&-((8\lambda_3^2-4\lambda_3-1)\lambda_2^2-(4\lambda_3^2-2\lambda_3+4)\lambda_2-(\lambda_3+4)\lambda_3+8)k_2^2k_3\\
&-R^2(\lambda_2-1)^2(8\lambda_2^2-4(\lambda_3-2)\lambda_2-(\lambda_3+4)\lambda_3+8)k_3,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{44} = {}& 12(\lambda_2-\lambda_3)^2(\lambda_2^2+\lambda_3^2+\lambda_3\lambda_2)k_1^4+12(\lambda_3-1)^2(\lambda_3^2+\lambda_3+1)k_2^4+12(\lambda_2-1)^2(\lambda_2^2+\lambda_2+1)k_3^4\\
&+3(8\lambda_3^4-4(\lambda_2+1)\lambda_3^3+8\lambda_2^2-(\lambda_2-1)^2\lambda_3^2-4(\lambda_2+1)\lambda_2\lambda_3)k_1^2k_2^2\\
&+3(8\lambda_2^4-4(\lambda_3+1)\lambda_2^3-(\lambda_3-1)^2\lambda_2^2+8\lambda_3^2-4(\lambda_3+1)\lambda_2\lambda_3)k_1^2k_3^2\\
&+3((8\lambda_3^2-4\lambda_3-1)\lambda_2^2-2(2\lambda_3^2-\lambda_3+2)\lambda_2-(\lambda_3+4)\lambda_3+8)k_2^2k_3^2\\
&+R^2(\lambda_2-\lambda_3)^2(8\lambda_2^2+8\lambda_3^2+(8\lambda_3-4)\lambda_2-4\lambda_3-1)k_1^2
\end{aligned}
$$

$$
\begin{aligned}
&- R^2(\lambda_3-1)^2(\lambda_2^2+4(\lambda_3+1)\lambda_2-8(\lambda_3^2+\lambda_3+1))k_2^2 \\
&+ R^2(\lambda_2-1)^2(8\lambda_2^2-4(\lambda_3-2)\lambda_2-(\lambda_3+4)\lambda_3+8)k_3^2 \\
&- 4R^4(\lambda_2^4-(\lambda_3+1)\lambda_2^3+\lambda_2^2\lambda_3-(\lambda_3-1)^2(\lambda_3+1)\lambda_2+(\lambda_3-1)^2(\lambda_3^2+\lambda_3+1)).
\end{aligned}
$$

Therefore, $\widetilde{SL(2,\mathbb{R})}\times\mathbb{R}$ or $SU(2)\times\mathbb{R}$ admit a Bach-flat left-invariant metric if and only if the structure constants in Equation (17) satisfy the equations $\{\mathfrak{P}_{ij}=0\}$. Let $\mathcal{I}\subset\mathbb{R}[\lambda_2,\lambda_3,k_1,k_2,k_3,R]$ be the ideal generated by the polynomials $\mathfrak{P}_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the graded reverse lexicographical order. A detailed analysis of the Gröbner basis shows that the polynomial

$$
\mathbf{g}_0=(\lambda_2-\lambda_3)k_1k_2^2k_3^2(k_2^2+k_3^2+R^2)(k_1^2+k_2^2+k_3^2+R^2) \tag{19}
$$

belongs to the basis. Since the zero sets of $\{\mathfrak{P}_{ij}=0\}$ and $\mathcal{I}=\langle\mathfrak{P}_{ij}\rangle=\langle\mathcal{G}\rangle$ coincide, we are led to the following cases:

$$
(1)\ \lambda_2=\lambda_3,\qquad (2)\ k_1=0,\qquad (3)\ k_2=0,\qquad (4)\ k_3=0.
$$

6.1. Case (1):

$\lambda_2=\lambda_3$. A direct calculation using Equation (18) implies that

$$
\mathfrak{P}_{14}=-3(\lambda_3-1)^2k_1(k_2^2+k_3^2)\lambda_3^2
$$

and therefore we have the following possibilities:

$$
(1.i)\ \lambda_3=1,\qquad (1.ii)\ k_1=0,\qquad (1.iii)\ k_2=k_3=0.
$$

6.1.1. Case (1.i):

$\lambda_2=\lambda_3$, $\lambda_3=1$. In this case we have $\lambda_1=\lambda_2=\lambda_3=1$ and a direct calculation shows that the corresponding Lie group given by Equation (17) is locally conformally flat, and thus a symmetric manifold [34].

6.1.2. Case (1.ii):

$\lambda_2=\lambda_3$, $k_1=0$. Computing a Gröbner basis of the ideal generated by $\mathcal{G}\cup\{\lambda_2-\lambda_3,k_1\}$ with respect to the graded reverse lexicographical order, we find that the polynomial

$$
\mathbf{g}_{12}=(\lambda_3-1)(k_2^2+k_3^2+R^2)^3R^2
$$

belongs to the ideal, leading to the solution $\lambda_3=1$ in Case (1.i) (§6.1.1).

6.1.3. Case (1.iii):

$\lambda_2=\lambda_3$, $k_2=k_3=0$. A direct calculation using Equation (18) shows that

$$
\mathfrak{P}_{44}=-4(\lambda_3-1)^2R^4
$$

and thus $\lambda_3=1$, which corresponds to Case (1.i) (§6.1.1).

6.2. Case (2):

$k_1=0$. Computing a Gröbner basis $\mathcal{G}_2$ of the ideal generated by $\mathcal{G}\cup\{k_1\}$ with respect to the graded reverse lexicographical order, one has that the polynomial

$$
\mathbf{g}_2=k_2k_3(\lambda_2-\lambda_3)(k_3^2+2R^2)(k_2^2+k_3^2+R^2)^2
$$

belongs to the basis. Since $\lambda_2=\lambda_3$ was solved in Case (1) (§6.1), we have the following possibilities:

$$
(2.i)\ k_2=0,\qquad (2.ii)\ k_3=0.
$$

6.2.1. Case (2.i):

$k_1 = 0$, $k_2 = 0$. We compute a Gröbner basis $\mathcal{G}_{21}$ of the ideal generated by $\mathcal{G}_2 \cup \{k_2\}$ with respect to the lexicographical order and we obtain that the polynomial $(\lambda_3 - 1)^2\lambda_3^4 R^6$ belongs to the basis. Since $\lambda_3 \neq 0$ the only possible solution is $\lambda_3 = 1$. Now, computing a new Gröbner basis $\mathcal{G}'_{21}$ of the ideal generated by $\mathcal{G}_{21} \cup \{\lambda_3 - 1\}$ with respect to the graded reverse lexicographical order we find that the polynomial $(\lambda_2 - 1)\lambda_2^2 R^4$ belongs to the basis. Thus we get the solution $\lambda_2 = \lambda_3 = 1$ which corresponds to Case (1.i) (§6.1.1).

6.2.2. Case (2.ii):

$k_1 = 0$, $k_3 = 0$. Considering the ideal $\mathcal{G}_2 \cup \{k_3\}$ and computing a Gröbner basis with respect to the lexicographical order, we find that the polynomial $k_2(\lambda_3 - 1)^2(k_2^2 + R^2)^3$ belongs to the basis. Since $k_2 = 0$ was treated in the previous case, we have $\lambda_3 = 1$, which together with $k_1 = k_3 = 0$ let us to get $\mathfrak{P}_{44} = -4(\lambda_2 - 1)^2\lambda_2^2 R^4$ from Equation (18). Hence, necessarily $\lambda_2 = \lambda_3 = 1$ and we are again in Case (1.i) (§6.1.1).

6.3. Case (3):

$k_2 = 0$. Computing a Gröbner basis $\mathcal{G}_3$ of the ideal generated by $\mathcal{G} \cup \{k_2\}$ with respect to the graded reverse lexicographical order, we find that the polynomial

$$\mathbf{g}_3 = (\lambda_2 - 1)(\lambda_3 - 1)k_3(k_3^2 + R^2)^2(k_1^2 + k_3^2 + R^2)R^2$$

belongs to the basis. Therefore we consider the following possibilities:

$$(3.i)\, \lambda_2 = 1, \qquad (3.ii)\, \lambda_3 = 1, \qquad (3.iii)\, k_3 = 0.$$

6.3.1. Case (3.i):

$k_2 = 0$, $\lambda_2 = 1$. Adding the polynomial $\lambda_2 - 1$ to $\mathcal{G}_3$ and computing a Gröbner basis with respect to the lexicographical order, we find that the polynomial $(\lambda_3 - 1)k_1^2(3k_3^2 + R^2)R^2$ belongs to the basis. Therefore, we are led to the previously considered Case (1) (§6.1) or Case (2) (§6.2).

6.3.2. Case (3.ii):

$k_2 = 0$, $\lambda_3 = 1$. Adding the polynomial $\lambda_3 - 1$ to $\mathcal{G}_3$ and computing a Gröbner basis with respect to the lexicographical order, we find that the polynomial $(\lambda_2 - 1)k_1 R^4$ belongs to the basis. Therefore, we are led to the previously considered Case (1) (§6.1) or Case (2) (§6.2).

6.3.3. Case (3.iii):

$k_2 = 0$, $k_3 = 0$. Adding the polynomial k_3 to $\mathcal{G}_3$ and computing a Gröbner basis with respect to the lexicographical order, we find in this case that the polynomial $(\lambda_3 - 1)^2\lambda_3^2 R^6$ belongs to the basis. This leads to Case (3.ii) (§6.3.2).

6.4. Case (4):

$k_3 = 0$. Computing a Gröbner basis $\mathcal{G}_4$ of the ideal generated by $\mathcal{G} \cup \{k_3\}$ with respect to the graded reverse lexicographical order, we find that the polynomial

$$\mathbf{g}_4 = k_1 k_2(\lambda_3 - 1)(k_2^2 + R^2)^2(k_1^2 + k_2^2 + R^2)(k_1^2 + k_2^2 + 4R^2)R^2$$

belongs to the basis. Since the cases $k_1 = 0$ and $k_2 = 0$ were already considered, one necessarily has $\lambda_3 = 1$. Using Equation (18), since $k_3 = 0$ and $\lambda_3 = 1$ we get $\mathfrak{P}_{24} = -3k_1^2 k_2(\lambda_2 - 1)^2$. Therefore, $\lambda_2 = 1 = \lambda_3$ and this leads again to Case (1) (§6.1), finishing the proof.

7. Proofs of Theorem 1.1, Theorem 1.2 and Corollary 1.3

Recall that if two Riemannian metrics are conformally equivalent, $\widetilde{g} = e^{2\sigma} g$, then their Weyl tensors of type $(1,3)$ coincide and thus $\widetilde{W} = e^{2\sigma} W$ for the Weyl tensors of type $(0,4)$. The converse does not hold in general, but it is true if the Weyl operator, viewed as a map $W\colon \Lambda^2 \to \Lambda^2$, has maximal rank (see [22]). Furthermore, if the conformal manifolds (M, g) and $(M, \widetilde{g})$ are both homogeneous, then $\|W\|^2$ and $\|\widetilde{W}\|^2$ are constant and, since $\|\widetilde{W}\|^2 = e^{-4\sigma}\|W\|^2$, either g and $\widetilde{g}$ are homothetic or otherwise both metrics are locally conformally flat. We will make extensively use of these facts to obtain the different homothety classes in Theorem 1.1.

As a matter of notation, for a given orthonormal basis $\{e_1, \dots, e_4\}$ on a Lie algebra $\mathfrak{g}$, we denote by $\{E_i^\pm\}$ the corresponding orthonormal basis of self-dual and anti-self-dual two-forms in $\Lambda^2_\pm(\mathfrak{g})$ given by $E_1^\pm = \frac{1}{\sqrt{2}}\left(e^1 \wedge e^2 \pm e^3 \wedge e^4\right)$, $E_2^\pm = \frac{1}{\sqrt{2}}\left(e^1 \wedge e^3 \mp e^2 \wedge e^4\right)$, and $E_3^\pm = \frac{1}{\sqrt{2}}\left(e^1 \wedge e^4 \pm e^2 \wedge e^3\right)$, where $\{e^i\}$ is the dual basis of $\{e_i\}$.

7.1. Proof of Theorem 1.1(i)

We consider the different Lie groups given by Lemma 5.1. Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be the left-invariant metric determined by Lemma 5.1(i). Considering the homothetic metric $\langle\,\cdot\,,\,\cdot\,\rangle^* = \frac{3}{8R^2}\langle\,\cdot\,,\,\cdot\,\rangle$, the Ricci operator of $\langle\,\cdot\,,\,\cdot\,\rangle^*$ takes the form $\operatorname{Ric} = -\operatorname{diag}[4,1,1,3]$ in the basis $\{e_1, \dots, e_4\}$. Moreover, the self-dual and anti-self-dual Weyl curvature operators become $W^\pm = \frac{1}{3}\operatorname{diag}[1,1,-2]$ in the induced basis of self-dual and anti-self-dual two-forms. Hence all metrics in Lemma 5.1(i) are homothetic. The expressions of $W^\pm$ show that the Weyl curvature operator has maximal rank. Hence, the necessary condition in Equation (4)–(ii) to be conformally Einstein is also sufficient. Let $\mathbb{T} \in \mathfrak{g}$ be an arbitrary vector and set $\mathbb{T} = \sum_k \mathbb{T}^k e_k$. Setting $R = 1$ and $h = \frac{1}{4}$, a straightforward calculation shows that $\mathfrak{C}(e_i, e_j, e_k) - W(e_i, e_j, e_k, \mathbb{T}) = 0$ if and only if $\mathbb{T} = 4\,e_4$. This shows that left-invariant metrics given by Lemma 5.1(i) are conformally Einstein.

Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be the left-invariant metric determined by Lemma 5.1(ii.c) and consider the homothetic metric $\langle\,\cdot\,,\,\cdot\,\rangle^* = \frac{6}{R^2}\langle\,\cdot\,,\,\cdot\,\rangle$. Then the Ricci operator of $\langle\,\cdot\,,\,\cdot\,\rangle^*$ takes the form $\operatorname{Ric} = -\operatorname{diag}[1,4,1,3]$ in the basis $\{e_1, \dots, e_4\}$, and the self-dual and anti-self-dual Weyl curvature operators become $W^\pm = \frac{1}{3}\operatorname{diag}[1,-2,1]$ in the induced basis of self-dual and anti-self-dual two-forms. Now one has that all metrics in Lemma 5.1(ii.c) are homothetic. Furthermore, $e_1 \mapsto e_2$ defines an orientation reversing homothety between the left-invariant metrics corresponding to cases (i) and (ii.c) in Lemma 5.1. One proceeds in a completely analogous way to show that all metrics in Lemma 5.1(ii.d) are homothetic and furthermore $e_1 \mapsto e_3$ defines an orientation reversing homothety between the left-invariant metrics corresponding to cases (i) and (ii.d) in Lemma 5.1. This completes the proof of Assertion (i) in Theorem 1.1.

Remark 7.1. The structure equations of $\mathfrak{g}_\alpha$: $[e_4, e_1] = e_1$, $[e_4, e_2] = \frac{1}{4}e_2 + \alpha e_3$, $[e_4, e_3] = -\alpha e_2 + \frac{1}{4}e_3$ are given in the dual basis $\{e^k\}$ by

$$de^4 = 0, \quad de^1 = e^1 \wedge e^4, \quad de^2 = \tfrac{1}{4}e^2 \wedge e^4 - \alpha e^3 \wedge e^4, \quad de^3 = \alpha e^2 \wedge e^4 + \tfrac{1}{4}e^3 \wedge e^4. \tag{20}$$

Integrating the expressions above gives coordinates (x, y, z, t) on $\mathbb{R}^4$ where

$$e^1 = e^{-t}dx, \quad e^2 = e^{-\frac{1}{4}t}(dy - \alpha z dt), \quad e^3 = e^{-\frac{1}{4}t}(dz + \alpha y dt), \quad e^4 = dt,$$

so that the metric expresses as

$$g_\alpha = e^{-2t}dx^2 + e^{-\frac{1}{2}t}(dy - \alpha z dt)^2 + e^{-\frac{1}{2}t}(dz + \alpha y dt)^2 + dt^2\,. \tag{21}$$

Now, a straightforward calculation shows that the conformal metric $\widetilde{g}_\alpha = e^{\frac{3}{2}t} g_\alpha$ is Ricci-flat.

The proof of Theorem 1.1(i) shows that the self-dual and anti-self-dual Weyl curvature operators have a distinguished eigenvalue with one-dimensional corresponding eigenspace. Hence $E_3^\pm$ define two-forms on $\mathbb{R}^4$. The structure equations (20) show that the underlying almost complex structures ($J^\pm e_1 = e_4$, $J^\pm e_2 = \pm e_3$) are integrable and moreover $dE_3^\pm = \theta \wedge E_3^\pm$ with $\theta = -\frac{1}{4}e^4$. Hence $(G_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle, J^\pm)$ is conformally Kähler and opposite-Kähler. Alternatively, results in [19] show that, since $\widetilde{g}_\alpha$ is Einstein and $\widetilde{W}^+ = \widetilde{W}^-$, the conformal metric $g^c_\alpha = (24\|\widetilde{W}^+\|^2)^{\frac{1}{3}}\widetilde{g}_\alpha$ is Kähler with respect to both orientations, where $\|\widetilde{W}^+\|^2 = \frac{3}{32}e^{-3t}$ in the coordinates (x, y, z, t) of Equation (21). Finally, observe that the Kähler metric g^c_α is locally a product $N \times \mathbb{R}^2$, where N is a warped product.

7.2. Proof of Theorem 1.1(ii)

Let $(G_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a half conformally flat Lie group given by Lemma 4.1(i) (see also Theorem 2.2). Following [17], let $\{e^k\}$ denote the dual basis of $\{e_k\}$ so that the structure equations are given by

$$de^4 = 0, \quad de^1 = e^1 \wedge e^4 + \alpha e^2 \wedge e^4, \quad de^2 = -\alpha e^1 \wedge e^4 + e^2 \wedge e^4, \quad de^3 = 2e^3 \wedge e^4 - e^1 \wedge e^2. \tag{22}$$

Integrating the expressions above gives coordinates (x, y, z, t) on $\mathbb{R}^4$ where (see [17])

$$e^1 = e^{-t}(dx{+}\alpha y dt), \quad e^2 = e^{-t}(dy{-}\alpha x dt), \quad e^3 = -e^{-2t}\left(dz + \frac{1}{2}(xdy - ydx) - \frac{1}{2}\alpha(x^2+y^2)dt\right), \quad e^4 = dt,$$

so that the metric expresses as

$$g_\alpha = e^{-2t}(dx + \alpha y dt)^2 + e^{-2t}(dy - \alpha x dt)^2 + e^{-4t}(dz + \frac{1}{2}(xdy - ydx) - \frac{1}{2}\alpha(x^2+y^2)dt)^2 + dt^2\,. \tag{23}$$

Now, a straightforward calculation shows that the conformal metric $\widetilde{g}_\alpha = e^{3t}g_\alpha$ is Ricci-flat, and thus $(G_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle)$ is conformally Einstein. This proves Assertion (ii) in Theorem 1.1.

Remark 7.2. A direct calculation shows that the Weyl tensor of $(G_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle)$ satisfies $W^+ = 0$ and $W^- = \operatorname{diag}[-2, 1, 1]$ with respect to the orthonormal basis $\{E_i^\pm\}$ of $\Lambda^2_\pm$. Hence, the distinguished eigenvalue of W^- with corresponding one-dimensional eigenspace defines a two-form E_1^- on G_α. The structure equations (22) show that the underlying almost complex structure ($J^- e_1 = e_2$, $J^- e_3 = -e_4$) is integrable and moreover $dE_1^- = \theta \wedge E_1^-$ with $\theta = e^4$. Hence $(G_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle, J')$ is conformally opposite-Kähler, since J' induces an opposite orientation on G_α. Alternatively, results in [19] show that, since $\widetilde{g}_\alpha$ is Einstein, the conformal metric $g^c_\alpha = (24\|\widetilde{W}^-\|^2)^{\frac{1}{3}}\widetilde{g}_\alpha$ is Kähler with respect to the opposite orientation, where $\|\widetilde{W}\|^2 = 6e^{-12t}$ in the coordinates (x, y, z, t) of Equation (23).

Finally, observe that since the conformal metric $(\mathbb{R}^4, \widetilde{g}_\alpha = e^{3t}g_\alpha)$ is Ricci-flat and anti-self-dual, it is pointwise Osserman [20]. Furthermore, for any unit vector field, the corresponding Jacobi operator $R_X = R(\,\cdot\,, X)X$ has eigenvalues $\mu = 0$, $\mu = -e^{-3t}$ and $\mu = \frac{1}{2}e^{-3t}$, the latter with multiplicity two. Since the non-zero eigenvalues are in a ratio $-1 : \frac{1}{2}$ they do not correspond to the eigenvalue structure of any globally Osserman manifold.

7.3. Proof of Theorem 1.1(iii)

Let $(\mathfrak{g}_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle_\alpha)$ be a Lie algebra given by Lemma 5.1(ii.a), and set

$$[e_4, e_1] = \frac{1}{R}e_1, \quad [e_4, e_2] = \frac{1}{R}(\alpha+1)^2 e_2, \quad [e_4, e_3] = \frac{1}{R}\alpha^2 e_3, \qquad \alpha > 0\,,$$

where $\{e_1, \dots, e_4\}$ is an orthonormal basis. Considering the homothetic metric $\langle\,\cdot\,,\,\cdot\,\rangle^*_\alpha = 2\frac{(\alpha^2+\alpha+1)^2}{R^2}\langle\,\cdot\,,\,\cdot\,\rangle_\alpha$, the Ricci operator of $\langle\,\cdot\,,\,\cdot\,\rangle^*$ and the self-dual and anti-self-dual Weyl curvature operators take the forms

$$\begin{aligned} \operatorname{Ric}_\alpha &= -\tfrac{1}{\alpha^2+\alpha+1}\operatorname{diag}[1, (\alpha+1)^2, \alpha^2, \alpha^2+\alpha+1]\,, \\ W^+_\alpha &= \tfrac{\alpha(\alpha+1)}{2(\alpha^2+\alpha+1)^2}\operatorname{diag}[\alpha, -(\alpha+1), 1] = W^-_\alpha\,, \end{aligned} \tag{24}$$

when expressed in the $\langle\,\cdot\,,\,\cdot\,\rangle^*_\alpha$-orthogonal basis $\{e_1, \dots, e_4\}$ and the induced basis of two-forms. Therefore $W^\pm_\alpha$ has three-distinct eigenvalues unless $\alpha = 1$.

In the special case $\alpha = 1$, considering the metric $\langle\cdot,\cdot\rangle^* = \frac{6}{R^2}\langle\cdot,\cdot\rangle_1$ one has that $\mathrm{Ric} = -\operatorname{diag}[1,4,1,3]$ and $W^+ = W^- = \frac{1}{3}\operatorname{diag}[1,-2,1]$. Hence, $e_1 \mapsto e_2$ defines an orientation reversing homothety with the metric in Lemma 5.1(i). Hence we assume $\alpha \neq 1$ in what follows. Furthermore, replacing α by α^{-1} in Equation (24) one has that $e_1 \mapsto e_3$ defines an orientation reversing homothety between the left-invariant metrics $\langle\cdot,\cdot\rangle_\alpha$ and $\langle\cdot,\cdot\rangle_{\alpha^{-1}}$. We therefore may assume $\alpha > 1$. Considering the homothetic metric $\langle\cdot,\cdot\rangle^*_\alpha$, a straightforward calculation shows that $\tau_\alpha = -3$ and $\|\rho_\alpha\|^2 = 3$. Moreover, the norm of the Weyl tensor satisfies $\|W_\alpha\|^2 = 4\frac{\alpha^2(\alpha+1)^2}{(\alpha^2+\alpha+1)^3}$. Hence two metrics $\langle\cdot,\cdot\rangle_\alpha$ and $\langle\cdot,\cdot\rangle_\beta$ with $\alpha,\beta \in (1,+\infty)$ are homothetic if and only if $\alpha^2(\alpha+1)^2(\beta^2+\beta+1)^3 = \beta^2(\beta+1)^2(\alpha^2+\alpha+1)^3$, and thus $\alpha = \beta$.

The necessary condition in Equation (4)-(ii) to be conformally Einstein is also sufficient in this case since by Equation (24) the Weyl tensor has maximal rank. Let $\mathbb{T} \in \mathfrak{g}_\alpha$ be an arbitrary vector and set $\mathbb{T} = \sum_k \mathbb{T}^k e_k$. A straightforward calculation shows that $\mathfrak{C}(e_i,e_j,e_k) - W(e_i,e_j,e_k,\mathbb{T}) = 0$ if and only if $\mathbb{T} = R(\alpha^2+\alpha+1)^{-1} e_4$. This shows that left-invariant metrics given by Lemma 5.1(ii.a) are conformally Einstein.

Let $(\mathfrak{g}_{\hat\alpha}, \langle\cdot,\cdot\rangle_{\hat\alpha})$ be a Lie algebra given by Lemma 5.1(ii.b), and set

$$[e_4,e_1] = \frac{1}{R}e_1, \quad [e_4,e_2] = \frac{1}{R}(\hat\alpha-1)^2 e_2, \quad [e_4,e_3] = \frac{1}{R}\hat\alpha^2 e_3, \quad \hat\alpha > 0, \quad \hat\alpha \neq 1\,,$$

where $\{e_1,\dots,e_4\}$ is an orthonormal basis. Let $\langle\cdot,\cdot\rangle^*_{\hat\alpha} = 2\frac{(\hat\alpha^2-\hat\alpha+1)^2}{R^2}\langle\cdot,\cdot\rangle_{\hat\alpha}$ be a homothetic metric. Then one has that $\tau_{\hat\alpha} = -3$ and moreover

$$\begin{array}{rcl} \mathrm{Ric}_{\hat\alpha} & = & -\frac{1}{\hat\alpha^2-\hat\alpha+1}\operatorname{diag}[1,(\hat\alpha-1)^2,\hat\alpha^2,\hat\alpha^2-\hat\alpha+1]\,, \\ W^+_{\hat\alpha} & = & \frac{\hat\alpha(\hat\alpha-1)}{2(\hat\alpha^2-\hat\alpha+1)^2}\operatorname{diag}[-\hat\alpha,\hat\alpha-1,1] = W^-_{\hat\alpha}\,. \end{array} \tag{25}$$

for the homothetic metric $\langle\cdot,\cdot\rangle^*_{\hat\alpha}$. Moreover it follows from the expressions in Equations (24) and (25) that $e_2 \mapsto e_3$ defines an orientation reversing isometry between the left-invariant metrics $\langle\cdot,\cdot\rangle^*_{\hat\alpha}$ and $\langle\cdot,\cdot\rangle^*_\alpha$ with $\alpha = \hat\alpha - 1$. This shows that any left-invariant metric in Lemma 5.1(ii.b) with $\hat\alpha > 1$ is homothetic to a left-invariant metrics in Lemma 5.1(ii.a) with $\alpha = \hat\alpha - 1$. Finally, observe that a replacement of $\hat\alpha$ by $\hat\alpha^{-1}$ in Equation (25) shows that for any $\hat\alpha \in (0,1)$ $e_1 \mapsto e_3$ defines a homothety between the metric $\langle\cdot,\cdot\rangle_{\hat\alpha}$ and $\langle\cdot,\cdot\rangle_{\hat\alpha^{-1}}$.

Finally, observe from (25) that $W^\pm_{\hat\alpha}$ have exactly three-distinct eigenvalues, unless $\hat\alpha = 2$ or $\hat\alpha = \frac{1}{2}$. Furthermore the discussion above shows that the cases $\hat\alpha = 2$ and $\hat\alpha = 2^{-1}$ are homothetic and moreover, they are homothetic to the case $\alpha = 1$ in Lemma 5.1(ii.a) previously considered. Therefore all metrics in Lemma 5.1(ii.b) are homothetic to the metrics in Lemma 5.1(ii.a) with $\alpha \geq 1$, which completes the proof of Assertion (iii) in Theorem 1.1.

Remark 7.3. The eigenvalue structure of the self-dual and anti-self-dual Weyl curvature tensors in Equation (24) shows that $\{E_i^+, E_i^-\}$, $i = 1,2,3$ define pairs of two-forms on G_α so that $E_i^+ \wedge E_i^- = 0$ and $E_i^+ \wedge E_i^+ = -E_i^- \wedge E_i^-$ for all $i = 1,2,3$. Furthermore, writing the structure equations of the Lie algebra $(\mathfrak{g}_\alpha, \langle\cdot,\cdot\rangle_\alpha)$ as

$$de^4 = 0, \quad de^1 = e^1\wedge e^4, \quad de^2 = (\alpha+1)^2 e^2\wedge e^4, \quad de^3 = \alpha^2 e^3 \wedge e^4, \tag{26}$$

one has $dE_i^\pm = \theta_i \wedge E_i^\pm$ with $\theta_1 = -(\alpha^2+2\alpha+2)e^4$, $\theta_2 = -(\alpha^2+1)e^4$ and $\theta_3 = -(2\alpha^2+2\alpha+1)e^4$. Therefore $\{E_i^+, E_i^-\}$ is a conformal symplectic pair on G_α for all $i = 1,2,3$ (see [7] for more information about symplectic pairs). In particular the six two-forms $E_i^\pm$ are conformally symplectic.

Furthermore, integrating the expressions in Equation (26) gives coordinates (x,y,z,t) on $\mathbb{R}^4$ where

$$e^1 = e^{-t}dx, \quad e^2 = e^{-(\alpha+1)^2 t}dy, \quad e^3 = e^{-\alpha^2 t}dz, \quad e^4 = dt,$$

so that the metric expresses as

$$g_\alpha = e^{-2t}dx^2 + e^{-2(\alpha+1)^2 t}dy^2 + e^{-2\alpha^2 t}dz^2 + dt^2\,. \tag{27}$$

As a consequence, $(\mathbb{R}^4, g_\alpha)$ has the structure of a multiply warped space of the form $\mathbb{R} \times_{f_1} \mathbb{R} \times_{f_2} \mathbb{R} \times_{f_3} \mathbb{R}$. Finally, a straightforward calculation shows that the conformal metric $\widetilde{g}_\alpha = e^{2(\alpha^2+\alpha+1)t} g_\alpha$ is Ricci-flat.

Remark 7.4. Bach-flat Kähler metrics are conformally Einstein [19]. Due to the conformal invariance of the Bach tensor (up to a functional factor), any Bach-flat conformally Kähler manifold is also conformally Einstein. The converse result is certainly not true. For instance, the eigenvalue structure of $W^\pm$ in Equation (24) shows that the homogeneous spaces corresponding to Theorem 1.1(iii) cannot be Kähler with respect to any conformal metric.

7.4. Proof of Theorem 1.2

Let us consider the two left-invariant metrics on $\mathbb{R}e_4 \ltimes E(1,1)$ at Lemma 3.1(i). The Lie brackets are given, with respect to an orthonormal basis $\{e_1, \dots, e_4\}$, by

$$[e_1, e_3] = (2+\sqrt{3})e_2, \quad [e_2, e_3] = e_1, \quad [e_4, e_1] = \varepsilon\sqrt{6+3\sqrt{3}}e_1, \quad [e_4, e_2] = \varepsilon\sqrt{6+3\sqrt{3}}e_2\,, \quad \varepsilon^2 = 1\,.$$

Now, an explicit calculation shows that the Ricci operator, in the basis $\{e_1, \dots, e_4\}$, takes the form $\mathrm{Ric} = -(2+\sqrt{3})\,\mathrm{diag}[6+\sqrt{3}, 6-\sqrt{3}, 3, 6]$.

Let $\{E_i^\pm\}$ be the corresponding orthonormal basis of self-dual and anti-self-dual two-forms given by $E_1^\pm = \frac{1}{\sqrt{2}}\left(e^1 \wedge e^2 \pm e^3 \wedge e^4\right)$, $E_2^\pm = \frac{1}{\sqrt{2}}\left(e^1 \wedge e^3 \mp e^2 \wedge e^4\right)$, and $E_3^\pm = \frac{1}{\sqrt{2}}\left(e^1 \wedge e^4 \pm e^2 \wedge e^3\right)$. Then, the self-dual and anti-self-dual Weyl curvature operators are given by

$$\begin{aligned} W^+_{(\varepsilon)} &= \tfrac{2+\sqrt{3}}{2}\,\mathrm{diag}[2, -1-3\varepsilon\sqrt{2}-\sqrt{3}, -1+3\varepsilon\sqrt{2}+\sqrt{3}]\,,\\ W^-_{(\varepsilon)} &= \tfrac{2+\sqrt{3}}{2}\,\mathrm{diag}[2, -1+3\varepsilon\sqrt{2}-\sqrt{3}, -1-3\varepsilon\sqrt{2}+\sqrt{3}]\,. \end{aligned}$$

Now, since the Weyl tensor has maximal rank, it follows from the results in [22] that the transformation $e_2 \mapsto -e_2$ defines an orientation reversing isometry between the two left-invariant metrics at Lemma 3.1(i).

Proceeding in a completely analogous way with the two left-invariant metrics at Lemma 3.1(ii),

$$[e_1, e_3] = (2-\sqrt{3})e_2, \quad [e_2, e_3] = e_1, \quad [e_4, e_1] = \varepsilon\sqrt{6-3\sqrt{3}}e_1, \quad [e_4, e_2] = \varepsilon\sqrt{6-3\sqrt{3}}e_2\,, \quad \varepsilon^2 = 1\,.$$

one has $\mathrm{Ric} = -(2-\sqrt{3})\,\mathrm{diag}[6-\sqrt{3}, 6+\sqrt{3}, 3, 6]$ and

$$\begin{aligned} W^+_{(\varepsilon)} &= \tfrac{2-\sqrt{3}}{2}\,\mathrm{diag}[2, -1-3\varepsilon\sqrt{2}+\sqrt{3}, -1+3\varepsilon\sqrt{2}-\sqrt{3}]\,,\\ W^-_{(\varepsilon)} &= \tfrac{2-\sqrt{3}}{2}\,\mathrm{diag}[2, -1+3\varepsilon\sqrt{2}+\sqrt{3}, -1-3\varepsilon\sqrt{2}-\sqrt{3}]\,. \end{aligned}$$

Again results in [22] show that the transformation $e_2 \mapsto -e_2$ defines an orientation reversing isometry between the two left-invariant metrics at Lemma 3.1(ii). Finally, let $\langle\,\cdot\,,\,\cdot\,\rangle_i$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{ii}$ be the left-invariant metrics corresponding to $\varepsilon = 1$ in the previous cases and consider the homothetic metric $\langle\,\cdot\,,\,\cdot\,\rangle^* = \frac{2-\sqrt{3}}{2+\sqrt{3}}\langle\,\cdot\,,\,\cdot\,\rangle_{ii}$. Considering the expressions above, it follows that $\{e_1 \mapsto e_2, e_2 \mapsto e_1\}$ defines an orientation reversing isometry between $\langle\,\cdot\,,\,\cdot\,\rangle_i$ and $\langle\,\cdot\,,\,\cdot\,\rangle^*$, thus showing that the four left-invariant metrics in Lemma 3.1 are homothetic.

Finally, observe that no metric in Lemma 3.1 is conformally Einstein. Indeed, considering the left-invariant metric $\langle\,\cdot\,,\,\cdot\,\rangle_i$ above, a straightforward calculation shows that, for any vector $\mathbb{T} \in \mathfrak{g}$ the necessary condition in Equation (4)-(ii) gives $\mathfrak{C}(e_1, e_2, e_3) - W(e_1, e_2, e_3, \mathbb{T}) = -3(5+3\sqrt{3}) \neq 0$, and thus $(G, \langle\,\cdot\,,\,\cdot\,\rangle_i)$ is strictly Bach-flat.

Consider the two left-invariant metrics on $\mathbb{R}e_4 \ltimes H^3$ at Lemma 4.1(ii.a). The Lie brackets are given, with respect to an orthonormal basis $\{e_1, \dots, e_4\}$, by

$$[e_1, e_2] = e_3, \quad [e_4, e_3] = -\varepsilon\frac{\sqrt{5}}{2\sqrt{2}}e_3\,, \quad [e_1, e_4] = \varepsilon\frac{1}{4}\sqrt{7+3\sqrt{5}}e_1, \quad [e_4, e_2] = \varepsilon\frac{1}{4}\sqrt{7-3\sqrt{5}}e_2\,, \quad \varepsilon^2 = 1\,.$$

Now, a explicit calculation shows that the Ricci operator, in the basis $\{e_1, \dots, e_4\}$, takes the form $\mathrm{Ric} = -\frac{3}{8}\,\mathrm{diag}[3+\sqrt{5}, 3-\sqrt{5}, 2, 4]$, and the self-dual and anti-self-dual Weyl curvature operators are given by

$$W^+_{(\varepsilon)} = -\tfrac{1}{8}\,\mathrm{diag}[2+\varepsilon\sqrt{10}, -1+\varepsilon\sqrt{7-3\sqrt{5}}, -1-\varepsilon\sqrt{7+3\sqrt{5}}]\,,$$
$$W^-_{(\varepsilon)} = -\tfrac{1}{8}\,\mathrm{diag}[2-\varepsilon\sqrt{10}, -1-\varepsilon\sqrt{7-3\sqrt{5}}, -1+\varepsilon\sqrt{7+3\sqrt{5}}]\,.$$

The left-invariant metrics at Lemma 4.1(ii.b), are given with respect to an orthonormal basis $\{e_1, \dots, e_4\}$ by

$$[e_1, e_2] = e_3, \quad [e_4, e_3] = \varepsilon\frac{\sqrt{5}}{2\sqrt{2}}e_3, \quad [e_1, e_4] = \varepsilon\frac{1}{4}\sqrt{7-3\sqrt{5}}e_1, \quad [e_4, e_2] = \varepsilon\frac{1}{4}\sqrt{7+3\sqrt{5}}e_2\,, \quad \varepsilon^2 = 1\,.$$

The Ricci operator, in the basis $\{e_1, \dots, e_4\}$, takes the form $\mathrm{Ric} = -\frac{3}{8}\,\mathrm{diag}[3-\sqrt{5}, 3+\sqrt{5}, 2, 4]$, and the self-dual and anti-self-dual Weyl curvature operators are given by

$$W^+_{(\varepsilon)} = -\tfrac{3}{24}\,\mathrm{diag}[2-\varepsilon\sqrt{10}, -1+\varepsilon\sqrt{7+3\sqrt{5}}, -1-\varepsilon\sqrt{7-3\sqrt{5}}]\,,$$
$$W^-_{(\varepsilon)} = -\tfrac{3}{24}\,\mathrm{diag}[2+\varepsilon\sqrt{10}, -1-\varepsilon\sqrt{7+3\sqrt{5}}, -1+\varepsilon\sqrt{7-3\sqrt{5}}]\,.$$

Hence the left-invariant metrics at Lemma 4.1(ii) are equivalent by an orientation preserving or reversing isometry.

In order to show that the left-invariant metrics in Lemma 4.1(ii) are strictly Bach-flat, we consider the necessary condition in Equation (4)-(ii) to be conformally Einstein. Let $\mathbb{T} \in \mathfrak{g}$ be an arbitrary vector and set $\mathbb{T} = \sum_k \mathbb{T}^k e_k$. Then one has

$$\mathfrak{C}(e_1, e_2, e_3) - W(e_1, e_2, e_3, \mathbb{T}) = \tfrac{1}{8}(-3+\sqrt{10}\mathbb{T}^4),$$
$$\mathfrak{C}(e_1, e_4, e_1) - W(e_1, e_4, e_1, \mathbb{T}) = \tfrac{1}{16}(3\sqrt{3+\sqrt{5}} - 2\mathbb{T}^4),$$

which are not compatible and thus the Lie group is strictly Bach-flat.

7.5. Proof of Corollary 1.3

A locally symmetric four-dimensional Bach-flat metric is either Einstein or locally conformally flat by Lemma 2.3. In the latter case (M, g) decomposes as a product $N^3(c) \times \mathbb{R}$ or $N^2(c) \times N^2(-c)$, where $N(c)$ is of constant curvature c. Clearly $N^3(c) \times \mathbb{R}$ is a rigid gradient Ricci soliton [33] while there are no Ricci solitons on $N^2(c) \times N^2(-c)$ (cf. Lemma 4.1 in [5]). Hence, in order to complete the proof, one needs to check all the possibilities in Theorem 1.1 and Theorem 1.2.

A straightforward calculation shows that half conformally flat Lie groups in Theorem 1.1(ii) are not algebraic Ricci solitons. Indeed, if $\mathfrak{D} = \mathrm{Ric} - \lambda\,\mathrm{Id}$ is a derivation then λ should satisfy the equations $\lambda + 6 = 0$ and $\lambda + \frac{3}{2} = 0$ which are incompatible.

Let $\mathfrak{g}_\alpha$ be a Lie algebra as in Theorem 1.1(i). Then a straightforward calculation shows that $\mathfrak{D} = \mathrm{Ric} + \frac{9}{8}\,\mathrm{Id}$ is a derivation and thus it defines an algebraic Ricci soliton. Analogously, Lie algebras $\mathfrak{g}_\alpha$ in Theorem 1.1(iii) are algebraic Ricci solitons, just considering the derivation $\mathfrak{D} = \mathrm{Ric} + 2(\alpha^2+\alpha+1)^2\,\mathrm{Id}$.

The Lie algebra corresponding to Theorem 1.2(i) is not an algebraic Ricci soliton since λ should satisfy the incompatible equations $\lambda - \sqrt{3} = 0$ and $\lambda + 12 + 7\sqrt{3} = 0$. On the contrary the Lie algebra $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{h}_3$ given at Theorem 1.2(ii) is an algebraic Ricci soliton, with $\mathfrak{D} = \mathrm{Ric} + \frac{3}{2}\,\mathrm{Id}$.

Remark 7.5. Rescaling the metrics in Corollary 1.3 so that the soliton constant takes the value $\lambda = -1$, one has that the scalar curvature $\tau = -3$ (thus $\tau \in (4\lambda, 0)$ as shown in [13]). Finally, observe that the Ricci solitons in Assertions (i) and (iii) are conformally Einstein.

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