A characterization of Kruskal sharing rules for minimum cost spanning tree problems

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Abstract

In Tijs et al. (2006) a new family of cost allocation rules is introduced in the context of cost spanning tree problems. In this paper we provide the first characterization of this family by means of population monotonicity and a property of additivity.

Keywords: minimum cost spanning tree problems, Kruskal's algorithm, sharing rules.

1 Introduction

Consider a group of agents demanding a particular service which is provided by a common supplier, called the source. Agents can be served through connections to the source, either directly or via other agents. Connections are costly. These situations are studied in the literature on "minimum cost spanning tree problems". Many real examples can be modeled in this way. For example, Bergantiños and Lorenzo (2004) studied a real situation where villagers had to pay the cost of constructing pipes from their respective houses to a water supply. Other examples are communication networks, such as telephone, Internet, wireless telecommunication, or cable television.

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The objective is to minimize the cost of connecting all agents to the source. This is achieved by a network of links that has no cycles. Such a network is called a "minimal cost spanning tree". Kruskal (1956) and Prim (1957) designed two algorithms for obtaining a minimal cost spanning tree. Once such a tree is obtained, its associated cost has to be divided among the agents. Bird (1976), Kar (2002), and Dutta and Kar (2004) introduced several rules for that purpose. Moreover, Bird (1976) associated with each minimum cost spanning tree problem a cooperative game with transferable utility. In this game, each coalition pays the minimum cost of connecting all of its members to the source, assuming that the agents outside the coalition are not present. Kar (2002) studied the Shapley value of this game whereas Granot and Huberman (1981 and 1984) studied the core and the nucleolus. Feltkamp et al. (1994) introduced the equal remaining obligation rule, which was studied by Bergantiños and Vidal-Puga (2005, 2007a, 2007b, and 2008). This rule belongs to a wide family of rules, introduced by Tijs et al. (2006), the family of "obligation rules". These rules are defined through Kruskal's algorithm and the philosophy of "construct and charge" (Moretti et al., 2005), *i.e.*, the minimal tree is built arc by arc and the cost of each arc is paid by all the agents who benefit from it. We refer to this family as the Kruskal family of sharing rules.

We provide the first characterization of the family. For it, we invoke two properties: population monotonicity and a suitable additivity property for this kind of problems. Population monotonicity was introduced by Thomson in the context of Bargaining theory (1983). The literature devoted to the analysis of this property in various models is surveyed in Thomson (1995).

The main result of this paper is not only important for the characterization itself, but it also provides us with an easy way to obtain the sharing functions associated with the rules. We also prove that a family of weighted Shapley rules are Kruskal sharing rules and we calculate the associated sharing functions.

The paper is organized as follows. In Section 2 we start with some preliminaries about minimum cost spanning tree problems. In Section 3 we characterize the Kruskal sharing rules. Section 4 connects the Kruskal sharing rules with solutions for cooperative games.

2 Minimum cost spanning tree problems

In this section we introduce minimum cost spanning tree problems.

Let $\mathcal{N} \subset \mathbb{N} = \{1, 2, \ldots\}$ be the set of all possible agents. Given a finite subset $N \subset \mathcal{N}$, an order π on N is a bijection $\pi : N \longrightarrow \{1, \ldots, |N|\}$ where, for each $i \in N$, $\pi(i)$ is the position of agent i. Let Π^N denote the set of all orders on N. Given $\pi \in \Pi^N$, $Pre(i, \pi)$ denotes the set of elements of N which come before i according to π, *i.e.*,

$$Pre(i, \pi) = \{j \in N \mid \pi(j) < \pi(i)\}.$$

Given $\pi \in \Pi^N$ and $S \subset N$, let π_S denote the order induced by π on S.

We deal with networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where N is the set of agents and 0 is a special node called the *source*. We consider $N = \{1, \ldots, n\}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ gives the cost of a direct link between any two nodes. We assume symmetric costs, *i.e.*, for each $i, j \in N_0$, $c_{ij} = c_{ji} \geq 0$ and for each $i \in N_0$, $c_{ii} = 0$.

We denote the set of all cost matrices with agent set N by $\mathcal{C}^N$. Given C, $C' \in \mathcal{C}^N$ we say that $C \leq C'$ if for each $i, j \in N_0$, $c_{ij} \leq c'_{ij}$.

A *minimum cost spanning tree problem*, briefly referred to as an *mcstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is a cost matrix. Given an *mcstp* (N_0, C) and $S \subset N$, we denote the restriction of the *mcstp* to $S_0 = S \cup \{0\}$ by (S_0, C).

A *network* g over N_0 is a subset of $\{(i, j) \mid i, j \in N_0, i \neq j\}$. The elements of g are called *arcs*. Since we assume symmetric costs, we work with undirected arcs, *i.e.*, $(i, j) = (j, i)$.

Given a network g and a pair of distinct nodes i and j, a *path from i to j in g* is a sequence of distinct arcs $g_{ij} = \{(i_{s-1}, i_s)\}_{s=1}^{p}$ that satisfy $(i_{s-1}, i_s) \in g$ for each $s \in \{1, 2, \ldots, p\}$, $i = i_0$ and $j = i_p$. A *cycle* is a path from i to i. Given $i, j \in N_0$, we say that i, j *are connected in* g if there exists a path from i to j.

A *tree* is a network such that for each $i \in N$, there is a unique path from i to the source.

We denote the set of all networks over N_0 by $\mathcal{G}^N$ and the set of networks over N_0 in such a way that every agent in N is connected to the source by $\mathcal{G}_0^N$.

Given a network g we say that $S \subset N_0$ is a *connected component* if two conditions hold. Firstly, for each $i, j \in S$, i and j are connected in g. Secondly, S is maximal, *i.e.*, for each $T \subset N_0$ with $S \subsetneq T$, there exist $i, j \in T$, $i \neq j$, such that i and j are not connected in g. Note that the set of connected components is a partition of N_0.

The following definitions appear in Norde et al. (2004). We say that $i, j \in S \subset N_0$,

$i \neq j$ are (C, S)-*connected* if there exists a path g_{ij} from i to j such that for each $(k, l) \in g_{ij}$, $k, l \in S$ and $c_{kl} = 0$. We say that $S \subset N_0$ is a C-*component* if two conditions hold. Firstly, for each $i, j \in S$, i and j are (C, S)-*connected*. Secondly, S is maximal, *i.e.*, for each $T \subset N_0$ with $S \subsetneq T$, there exist $i, j \in T$, $i \neq j$, such that i and j are not (C, T)-*connected*. The set of C-*components* is a partition of N_0 (Norde et al., 2004).

Every *mcstp* can be written as a non-negative combination of *mcstp* in which the costs of the arcs are 0 or 1 (Norde et al., 2004). The next lemma states this result in a slightly different but equivalent way, using our notation.

Lemma 1 *For each mcstp* (N_0, C), *there exists a family* $\{C^q\}_{q=1}^{m(C)}$ *of cost matrices and a family* $\{x^q\}_{q=1}^{m(C)}$ *of non-negative real numbers satisfying three conditions:*

(1) $C = \sum_{q=1}^{m(C)} x^q C^q$.

(2) *For each* $q \in \{1, \ldots, m(C)\}$, *there exists a network* g^q *such that* $c_{ij}^q = 1$ *if* $(i, j) \in g^q$ *and* $c_{ij}^q = 0$ *otherwise.*

(3) *Let* $q \in \{1, \ldots, m(C)\}$ *and* $\{i, j, k, l\} \subset N_0$. *If* $c_{ij} \leq c_{kl}$, *then* $c_{ij}^q \leq c_{kl}^q$.

Given an *mcstp* (N_0, C) and $g \in \mathcal{G}^N$, we define the *cost* of g as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there is no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimal tree* for (N_0, C), briefly referred to as an *mt*, is a tree $t \in \mathcal{G}_0^N$ such that $c(t) = \min_{g \in \mathcal{G}_0^N} c(g)$. An *mt* always exists, although it may not be unique. Given an *mcstp* (N_0, C), $m(N_0, C)$ denotes the cost of any *mt* t in (N_0, C).

Given an *mcstp* (N_0, C) and an *mt* t, the *minimal network* (N_0, C^t) associated with t is defined as follows (Bird, 1976): $c_{ij}^t = \max_{(k,l) \in g_{ij}} \{c_{kl}\}$, where g_{ij} denotes the unique path in t from i to j. The same minimal network is obtained if we consider a different *mt* for the original *mcstp* (Aarts and Driessen, 1993).

The *irreducible form* of an *mcstp* (N_0, C) is defined as the minimal network $(N_0, C^*) = (N_0, C^t)$ associated with a particular *mt* t (Bergantiños and Vidal-Puga, 2007b). If (N_0, C^*) is an irreducible form, we say that C^* is an *irreducible matrix*. Moreover, $C^* \leq C$. Note that a matrix is irreducible if reducing the cost of any arc, the cost of connecting agents to the source is also reduced.

After obtaining an *mt*, one of the most important issues addressed in the literature on *mcstp* is how to divide its cost $m(N_0, C)$ among the agents.

A *cost allocation rule* is a map ψ that associates with each *mcstp* (N_0, C) a vector $\psi(N_0, C) \in \mathbb{R}^N$ such that $\sum_{i \in N} \psi_i(N_0, C) = m(N_0, C)$. Given an agent i, $\psi_i(N_0, C)$ denotes its payment.

3 A characterization of Kruskal sharing rules

Kruskal sharing rules are defined following Kruskal's algorithm (1956). The idea behind this algorithm is to construct a tree by sequentially adding arcs with the lowest cost without introducing cycles. Formally, Kruskal's algorithm is defined as follows.

We start with $A^0(C) = \{(i, j) \mid i, j \in N_0, i \neq j\}$ and $g^0(C) = \emptyset$.

Stage 1: Let $(i, j) \in A^0(C)$ be an arc such that $c_{ij} = \min_{(k,l) \in A^0(C)} \{c_{kl}\}$. (If there are several arcs satisfying this condition, select just one). We have that
$\left(i^1(C), j^1(C)\right) = (i, j)$, $A^1(C) = A^0(C) \setminus \{(i, j)\}$, and $g^1(C) = \left\{\left(i^1(C), j^1(C)\right)\right\}$.

Stage p+1. We have defined the sets $A^p(C)$ and $g^p(C)$. Let $(i, j) \in A^p(C)$ be an arc such that $c_{ij} = \min_{(k,l) \in A^p(C)} \{c_{kl}\}$. (If there are several arcs satisfying this condition, select just one). Two cases are possible:

1. $g^p(C) \cup \{(i, j)\}$ contains a cycle. **Go to the beginning of Stage p+1** with $A^p(C) = A^p(C) \setminus \{(i, j)\}$ and $g^p(C)$ the same.

2. $g^p(C) \cup \{(i, j)\}$ has no cycles. Set $(i^{p+1}(C), j^{p+1}(C)) = (i, j)$, $A^{p+1}(C) = A^p(C) \setminus \{(i, j)\}$, and $g^{p+1}(C) = g^p(C) \cup \{(i^{p+1}(C), j^{p+1}(C))\}$. **Go to Stage p+2.**

This algorithm is completed in n stages. It leads to a tree, which may not be unique. We say that a minimal tree $g^n(C)$ obtained at the end of Step n in the Kruskal algorithm is a **Kruskal tree**.

When there is no ambiguity, we write A^p, g^p, and (i^p, j^p) instead of $A^p(C)$, $g^p(C)$, and $(i^p(C), j^p(C))$, respectively.

Given a network g, let $P(g) = \{T_k(g)\}_{k=1}^{n(g)}$ denote the unique partition of N_0 in connected components induced by g. Formally,

- If $i, j \in T_k(g)$, i and j are connected in g.

- If $i \in T_k(g)$, $j \in T_l(g)$ and $k \neq l$, i and j are not connected in g.

Given a network g and $i \in N_0$, let $S(P(g), i)$ denote the element of $P(g)$ to which i belongs.

Tijs et al. (2006) introduce Kruskal sharing rules for *mcstp*. We present this definition in a different but equivalent way, using our notation.

Let $N \subset \mathcal{N}$ and $S \subset N_0$. A *sharing function*, o, is a map defined as follows:

- if $0 \in S$, for each $i \in S \setminus \{0\}$, $o_i(S) = 0$.
- if $0 \notin S$, $o(S) \in \Delta(S) = \{x \in \mathbb{R}^S_+ : \sum_{i \in S} x_i = 1\}$ and for each $S \subset T$, and each $i \in S$, $o_i(T) \leq o_i(S)$.

We can associate a *Kruskal sharing rule* ϕ^o with each sharing function o. The idea is as follows. At each stage of Kruskal's algorithm an arc (i^p, j^p) is added to the network. The cost of this arc is paid by the agents involved in the connected component to which the agents i^p, j^p belong, except for those who were connected to the source before its construction. Each of these agents pays the difference between his share before the arc is added to the network and after it is added.

We now define Kruskal sharing rules formally. Given an *mcstp* (N_0, C), let g^n be a Kruskal tree. For each $i \in N$,

$$\phi^o_i(N_0, C) = \sum_{p=1}^{n} c_{i^p j^p}(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i)))$$

Note that, from the definition of Kruskal sharing rules, it is not clear that ϕ^o is a cost allocation rule for *mcstp*. For instance, ϕ^o could depend on the selected Kruskal tree. Tijs et al. (2006) proved that each Kruskal sharing rule ϕ^o is a well-defined rule.

In the next example we calculate the family of Kruskal sharing rules related to an *mcstp*. In this example, there are two different Kruskal trees.

Example 1 *Consider the mcstp* (N_0, C) *described in Figure 1.*

Following Kruskal's algorithm, we choose, as it is shown in the figure, the mt $\{(1,2), (2,3), (0,1)\}$.

In the next table we describe the quantity assigned to each agent at each step.

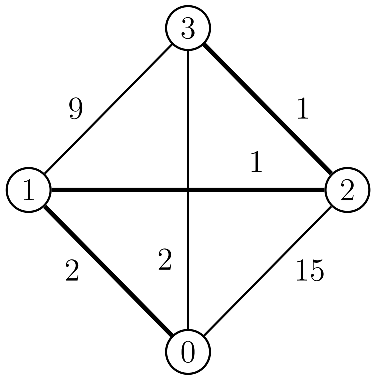

Figure 1: *mcstp* (N_0, C)

Stage	*Arc*	*Agent* 1	*Agent* 2	*Agent* 3
1	$(1,2)$	$1(o_1(\{1\}) - o_1(\{1,2\}))$	$1(o_2(\{2\}) - o_2(\{1,2\}))$	0
2	$(2,3)$	$1(o_1(\{1,2\}) - o_1(N))$	$1(o_2(\{1,2\}) - o_2(N))$	$1(o_3(\{3\}) - o_3(N))$
3	$(0,1)$	$2(o_1(N) - o_1(N_0))$	$2(o_2(N) - o_2(N_0))$	$2(o_3(N) - o_3(N_0))$

Finally, we obtain

$$\phi^o(N_0, C) = (1 + o_1(N), 1 + o_2(N), 1 + o_3(N)),$$

with $o_1(N)$, $o_2(N)$, $o_3(N) \geq 0$ *and* $o_1(N) + o_2(N) + o_3(N) = 1$.

Note that if we consider the other Kruskal tree, given by $\{(2,3), (1,2), (0,1)\}$*, we obtain the same result because Kruskal sharing rules are well-defined (Tijs et al., 2006).*

Next, we give the first characterization of the family of Kruskal sharing rules. This characterization is based on a property of monotonicity over the set of agents and a property of additivity defined in Bergantiños and Vidal-Puga (2008). This result holds for any set of possible agents $\mathcal{N}$ except for two-agent sets. In this situation, it is sufficient to add the property of non-negativity.

Population monotonicity (PM): For each *mcstp* (N_0, C), each $S \subset N$, and each $i \in S$

$$\psi_i(S_0, C) \geq \psi_i(N_0, C).$$

This property implies that if some agents leave no remaining agent should be better off than before.

Additivity is a standard property and it has been used in many situations. In the case of *mcstp*, additivity says that if we have two *mcstp* (N_0, C) and (N_0, C') then, $\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C')$. The following example shows that no rule satisfies this property.

Example 2 *Consider the mcstp* (N_0, C) *and* (N_0, C') *described in Figures 2 and 3.*

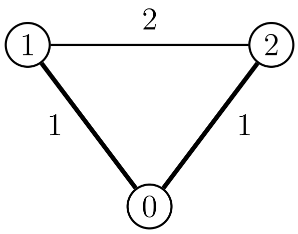

Figure 2: *mcstp* (N_0, C)

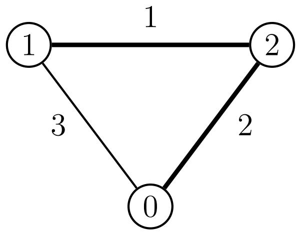

Figure 3: *mcstp* (N_0, C')

In this case $m(N_0, C) = 2$ *and* $m(N_0, C') = 3$, *while* $m(N_0, C+C') = 6$. *So, no rule satisfies additivity.*

For this reason Bergantiños and Vidal-Puga (2008) introduce the constrained additivity property. In order to define this property we need to introduce the concept of *similar problems.*

The *mcstp* (N_0, C) and (N_0, C') are *similar* if there exists an *mt* $t = \{(i^0, i)\}_{i \in N}$ in (N_0, C), (N_0, C'), and $(N_0, C + C')$ and an order $\pi = (i_1, \ldots, i_n) \in \Pi^N$ such that $c_{i_1^0 i_1} \leq c_{i_2^0 i_2} \leq \ldots \leq c_{i_n^0 i_n}$ and $c'_{i_1^0 i_1} \leq c'_{i_2^0 i_2} \leq \ldots \leq c'_{i_n^0 i_n}$, *i.e.*, the arcs in the *mt* t are ordered in the same way in both problems. Note that two similar problems share at least one *mt*.

Constrained additivity (CA): For each pair of similar *mcstp* (N_0, C) and (N_0, C'), we have

$$\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C').$$

From a mathematical point of view, CA is an appealing property because if a rule is additive the initial problem can be decomposed in a sum of simpler problems which are usually easier to solve. So, an additive rule is easier to compute. Besides, in many problems it is possible to characterize rules with additivity and very "basic" properties. For example, the Shapley value (Shapley, 1953b), one of the most important solutions for games with transferable utility, is characterized by means of additivity, efficiency, symmetry, and dummy player. There are many values satisfying efficiency, symmetry, and dummy player, for example the nucleolus (Schmeidler, 1969), but the Shapley value is the only one which satisfies additivity.

Moreover, given an *mcstp* (N_0, C), assume that some additional costs that were not considered in the initial problem appear. Besides, assume that the *mcstp* associated with these extra costs is similar to (N_0, C). Then, CA says that the cost allocation provided by the rule should be the same if the problem is reevaluated considering these extra costs or if we sum up the initial allocation and the allocation of these extra costs.

Non-negativity (NN): For each *mcstp* (N_0, C) and each $i \in N$, $\psi_i(N_0, C) \geq 0$.

Below, we introduce two interesting properties, which are also satisfied by Kruskal sharing rules.

Strong Cost Monotonicity (SCM): For each pair of *mcstp* (N_0, C) and (N_0, C') such that $C \leq C'$,

$$\psi(N_0, C) \leq \psi(N_0, C').$$

This property implies that if some connection costs increase, no agent ends up better off. It was introduced by Bergantiños and Vidal-Puga (2007b). Tijs et al. (2006) proved that Kruskal sharing rules satisfy SCM.

Continuity (CON): ψ is a continuous function of C.

Lemma 2 *Consider an mcstp (N_0, C) and $\{C^q\}_{q=1}^{m(C)}$ and $\{x^q\}_{q=1}^{m(C)}$ satisfying the conditions of Lemma 1. Then, the mcstp $\{(N_0, x^q C^q)\}_{q=1}^{m(C)}$ are similar.*

Proof. Suppose that there exists an *mt* $t = \{(i^0, i)\}_{i \in N}$ in (N_0, C) and assume, without loss of generality, that $c_{1^0 1} \leq c_{2^0 2} \leq \ldots \leq c_{n^0 n}$. By Lemma 1 (3), t is an *mt* in (N_0, C^q) and for each $q = 1, \ldots, m(C)$, $c^q_{1^0 1} \leq c^q_{2^0 2} \leq \ldots \leq c^q_{n^0 n}$. Since $x^q \geq 0$, t is also an *mt* in $(N_0, x^q C^q)$ and for each $q = 1, \ldots, m(C)$, $x^q c^q_{1^0 1} \leq x^q c^q_{2^0 2} \leq \ldots \leq x^q c^q_{n^0 n}$. ■

Proposition 1 *Kruskal sharing rules satisfy NN, PM, CA, and CON.*

Proof.

Kruskal sharing rules satisfy NN by definition.

Kruskal sharing rules satisfy PM (Tijs et al., 2006).

To show that Kruskal sharing rules satisfy CA, we consider two similar *mcstp* (N_0, C) and (N_0, C'). Assume, without loss of generality, that $t = \{(i^0, i)\}_{i \in N}$ is a common *mt* for both *mcstp* such that $c_{1^0 1} \leq c_{2^0 2} \leq \ldots \leq c_{n^0 n}$ and $c'_{1^0 1} \leq c'_{2^0 2} \leq \ldots \leq c'_{n^0 n}$. Note that $t = \{(i^0, i)\}_{i \in N}$ is also an *mt* in $(N_0, C + C')$ and $c_{1^0 1} + c'_{1^0 1} \leq c_{2^0 2} + c'_{2^0 2} \leq \ldots \leq$

$c_{n^0n} + c'_{n^0n}$. Furthermore, Kruskal sharing rules are independent of the chosen Kruskal tree (Tijs et al., 2006). Therefore, given a Kruskal sharing rule ϕ^o,

$$\begin{aligned}
\phi_i^o(N_0, C+C') &= \sum_{p=1}^{n}(c_{p^0p} + c'_{p^0p})(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i))) \\
&= \sum_{p=1}^{n} c_{p^0p}(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i))) \\
&+ \sum_{p=1}^{n} c'_{p^0p}(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i))) \\
&= \phi_i^o(N_0, C) + \phi_i^o(N_0, C').
\end{aligned}$$

In the case of CON, we define for each *mcstp* (N_0, C) and each $\epsilon > 0$, the *mcstp* $(N_0, C^{+\epsilon})$ and $(N_0, C^{-\epsilon})$, where for each $i, j \in N_0$, $c_{ij}^{+\epsilon} = c_{ij} + \epsilon$ and $c_{ij}^{-\epsilon} = \max\{0, c_{ij} - \epsilon\}$. Given an *mt* $t = \{(i^0, i)\}_{i \in N}$ for (N_0, C) such that $c_{1^01} \leq c_{2^02} \leq \ldots \leq c_{n^0n}$, t is also an *mt* for $(N_0, C^{+\epsilon})$ and $(N_0, C^{-\epsilon})$. Moreover, $c_{1^01}^{+\epsilon} \leq c_{2^02}^{+\epsilon} \leq \ldots \leq c_{n^0n}^{+\epsilon}$ and $c_{1^01}^{-\epsilon} \leq c_{2^02}^{-\epsilon} \leq \ldots \leq c_{n^0n}^{-\epsilon}$.

The allocation generated by the Kruskal sharing rule in the *mcstp* $(N_0, C^{+\epsilon})$ is given by

$$\begin{aligned}
\phi_i^o\left(N_0, C^{+\epsilon}\right) &= \sum_{p=1}^{n} c_{p^0p}^{+\epsilon}(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i))) \\
&= \sum_{p=1}^{n}(c_{p^0p} + \epsilon)(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i))) \\
&= \sum_{p=1}^{n} c_{p^0p}(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i))) + \epsilon \\
&= \phi_i^o(N_0, C) + \epsilon.
\end{aligned}$$

On the other hand, the allocation generated by the Kruskal sharing rule in the *mcstp*

$(N_0, C^{-\epsilon})$ is given by

$$\begin{aligned}
\phi_i^o\left(N_0, C^{-\epsilon}\right) &= \sum_{p=1}^{n} c_{p^0p}^{-\epsilon}(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i))) \\
&= \sum_{p=1}^{n} \max\{0, c_{p^0p} - \epsilon\}(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i))) \\
&\geq \sum_{p=1}^{n} c_{p^0p}(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i))) - \epsilon \\
&= \phi_i^o(N_0, C) - \epsilon.
\end{aligned}$$

Consider now the sequence of cost matrices $\{C^\epsilon\}$ such that for each $i, j \in N_0$, $|c_{ij}^\epsilon - c_{ij}| < \epsilon$. Note that $C^{-\epsilon} \leq C^\epsilon \leq C^{+\epsilon}$. Since Kruskal sharing rules satisfy SCM (Tijs et al., 2006),

$$\phi_i^o(N_0, C) - \epsilon \leq \phi_i^o\left(N_0, C^{-\epsilon}\right) \leq \phi_i^o\left(N_0, C^\epsilon\right) \leq \phi_i^o\left(N_0, C^{+\epsilon}\right) = \phi_i^o(N_0, C) + \epsilon.$$

Thus, for each $i \in N$, $|\phi_i^o\left(N_0, C^\epsilon\right) - \phi_i^o(N_0, C)| < \epsilon$. ■

Theorem 1 *Suppose that $|\mathcal{N}| \geq 3$. A rule ψ satisfies PM and CA if and only if for each mcstp (N_0, C) such that $N \subset \mathcal{N}$,*

$$\psi(N_0, C) = \phi^{\hat{o}}(N_0, C),$$

where $\hat{o}$ is the sharing function defined, for each $S \in 2^N \setminus \{\emptyset\}$, by

$$\hat{o}(S) = \psi(S_0, \widehat{C})$$

and the cost matrix $\widehat{C} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$.

Proof.

Existence. Proposition 1 proves that Kruskal sharing rules satisfy PM and CA.

Uniqueness. Consider a rule ψ satisfying PM and CA. We divide the proof in several claims.

Claim 1 *$\hat{o}$ is a sharing function.*

Proof of Claim 1.

As ψ satisfies PM, for each $S \subset T \in 2^N \setminus \{\emptyset\}$ and each $i \in S$, $\hat{o}_i(T) \leq \hat{o}_i(S)$.

Moreover, since for each $S \in 2^N \setminus \{\emptyset\}$, $m(S_0, \widehat{C}) = 1$, if for each $S \in 2^N \setminus \{\emptyset\}$ and each $i \in S$, $\hat{o}_i(S) \geq 0$, then the vector $\hat{o}(S)$ belongs to the simplex in $\mathbb{R}^S$.

Suppose that $S = \{i\}$, with $i \in N$. We know that $\hat{o}_i(S) = \psi_i(S_0, \widehat{C}) = 1$. Next, suppose that $|S| > 1$. In this case, by PM, for each $i \in S$ and each $j \in S \setminus \{i\}$, $\psi_j(S_0, \widehat{C}) \leq \psi_j((S_0 \setminus \{i\}), \widehat{C})$. Thus, $1 - \psi_i(S_0, \widehat{C}) \leq \sum_{j \in S \setminus \{i\}} \psi_j(S_0 \setminus \{i\}, \widehat{C}) = 1$ and, hence, $\hat{o}_i(S) = \psi_i(S_0, \widehat{C}) \geq 0$. ■

Consider $C = \sum_{q=1}^{m(C)} x^q C^q$ with $\{C^q\}_{q=1}^{m(C)}$ and $\{x^q\}_{q=1}^{m(C)}$ satisfying the conditions of Lemma 1. Since ψ satisfies CA, by Lemma 2, $\psi(N_0, C) = \sum_{q=1}^{m(C)} \psi(N_0, x^q C^q)$. By Proposition 1, Kruskal sharing rules satisfy CA. Thus, $\phi^o(N_0, C) = \sum_{q=1}^{m(C)} \phi^o(N_0, x^q C^q)$. Therefore, it is sufficient to prove that $\psi(N_0, x^q C^q) = \phi^{\hat{o}}(N_0, x^q C^q)$.

Claim 2 *Consider a sharing function o and an mcstp (N_0, C) such that there exists a network g with $c_{ij} = x \geq 0$ if $(i,j) \in g$ and $c_{ij} = 0$ otherwise. Let $\{T_r\}_{r=1}^m$ be the partition of N_0 in C-components. Then, for each $i \in T_r$ and each $r = 1, \ldots, m$,*

$$\phi_i^o(N_0, C) = \begin{cases} 0, & 0 \in T_r \\ x o_i(T_r), & 0 \notin T_r. \end{cases}$$

Proof of Claim 2. Given a sharing function o, let us consider the Kruskal sharing rule ϕ^o. If we apply Kruskal's algorithm, we assume that in the first $n - m$ stages the agents in each component are connected to one another, *i.e.*, $P(g^{n-m}) = \{T_r\}_{r=1}^m$. Since for each $p = 1, \ldots, n-m$, $c_{i^p j^p} = 0$ and $o_i(T) = 0$ when the source is in T, we distinguish two cases:

1. $0 \in T_r$. In this case, for each $i \in T_r$, $\phi_i^o(N_0, C) = 0$.

2. $0 \notin T_r$. At Stage $n - m + 1$ of Kruskal's algorithm, it is possible to select the arc (i^{n-m+1}, j^{n-m+1}) such that $i^{n-m+1} \in T_r$ and $j^{n-m+1} = 0$. Therefore, for each $i \in T_r$,
$$\phi_i^o(N_0, C) = c_{i^{n-m+1}0} o_i(S(P(g^{n-m}), i)) = x o_i(T_r).$$

■

Claim 3 *Given a mcstp* (N_0, C) *such that there exists a network* g *with* $c_{ij} = x \geq 0$ *if* $(i,j) \in g$ *and* $c_{ij} = 0$ *otherwise,* $\phi^{\hat{o}}(N_0, C) = \psi(N_0, C)$.

Proof of Claim 3. Since $m(N_0, C) = \sum_{r=1}^{m} m((T_r)_0, C)$ (Bergantiños and Vidal-Puga, 2008) and ψ satisfies PM, for each $i \in T_r$ and each $r = 1, \ldots, m$, $\psi_i(N_0, C) \leq \psi_i((T_r)_0, C)$. Moreover, $\sum_{i \in T_r} \psi_i((T_r)_0, C) = m((T_r)_0, C)$ and $\sum_{i \in N} \psi_i(N_0, C) = m(N_0, C)$. Thus, for each $i \in T_r$ and each $r = 1, \ldots, m$, $\psi_i(N_0, C) = \psi_i((T_r)_0, C)$.

We define two cost matrices $\tilde{C}$ and $\bar{C}$ by

$$\tilde{c}_{ij} = \begin{cases} 0 & \text{if } 0 \in \{i,j\} \\ c_{ij} & \text{otherwise} \end{cases} \text{ and } \bar{c}_{ij} = \begin{cases} c_{ij} & \text{if } 0 \in \{i,j\} \\ 0 & \text{otherwise} \end{cases} \text{ for each } i, j \in N_0.$$

Note that $((T_r)_0, \tilde{C})$ and $((T_r)_0, \bar{C})$ are similar and that $C = \tilde{C} + \bar{C}$. By CA,

$$\psi((T_r)_0, C) = \psi((T_r)_0, \tilde{C}) + \psi((T_r)_0, \bar{C}).$$

Since for each $i \in T_r$ and each $r = 1, \ldots, m$, $m((T_r)_0, \tilde{C}) = 0$ and $m(\{i\}_0, \tilde{C}) = 0$, by PM, for each $i \in T_r$ and each $r = 1, \ldots, m$, $\psi_i((T_r)_0, \tilde{C}) = \psi_i(\{i\}_0, \tilde{C}) = 0$. Therefore, for each $r = 1, \ldots, m$, $\psi((T_r)_0, C) = \psi((T_r)_0, \bar{C})$.

We distinguish two cases:

Case 1. $0 \in T_r$. We have to prove that for each $i \in T_r$, $\psi_i(N_0, C) = 0 = \phi_i^{\hat{o}}(N_0, C)$.

We distinguish two subcases:

Subcase 1.a. For each $i \in T_r$, $c_{i0} = 0$. In this case, for each $i \in T_r$, $\psi_i(N_0, C) = \psi_i((T_r)_0, C) = 0 = \phi_i^{\hat{o}}(N_0, C)$.

Subcase 1.b. There exist $j, k \in T_r$ such that $c_{0j} = 0$ and $c_{0k} = x$.

Following similar arguments to Bergantiños and Vidal-Puga (2008), we consider $T_r^1 = \{i \in T_r : c_{0i} = x\} \cup \{j\}$ and $T_r^2 = \{i \in T_r : c_{0i} = 0\} \setminus \{j\}$.

We know that $m((T_r)_0, \bar{C}) = m((T_r^1)_0, \bar{C}) + m((T_r^2)_0, \bar{C})$. By PM,

$$\psi_i(N_0, C) = \psi_i((T_r)_0, \bar{C}) = \begin{cases} \psi_i((T_r^1)_0, \bar{C}) & \text{if } i \in T_r^1 \\ \psi_i((T_r^2)_0, \bar{C}) & \text{if } i \in T_r^2 \end{cases}$$

By Subcase 1.a., for each $i \in {T_r}^2$, $\psi_i((T_r^2)_0, \bar{C}) = 0$.

By CA, $\psi((T_r^1)_0, \bar{C}) = \sum_{i \in T_r^1 \setminus \{j\}} \psi((T_r^1)_0, (\bar{C})^i)$ where $(\bar{c}_{0i})^i = x$ and $(\bar{c}_{kl})^i = 0$ otherwise.

Since $m((T_r^1)_0, (\bar{C})^i) = m(\{i,j\}_0, (\bar{C})^i) + \sum_{k \in T_r^1 \setminus \{i,j\}} m(\{k\}_0, (\bar{C})^i)$, by PM,

$$\psi_k((T_r^1)_0, (\bar{C})^i) = 0 \text{ for each } k \in T_r^1 \setminus \{j, i\},$$
$$\psi_j((T_r^1)_0, (\bar{C})^i) = \psi_j(\{i,j\}_0, (\bar{C})^i), \text{ and}$$
$$\psi_i(({T_r}^1)_0, (\bar{C})^i) = \psi_i(\{i,j\}_0, (\bar{C})^i).$$

If $\psi_j((T_r^1)_0, (\bar{C})^i) = 0$ and $\psi_i((T_r^1)_0, (\bar{C})^i) = 0$, then, for each $i \in T_r^1$, $\psi_i(N_0, C) = \psi_i((T_r)_0, \bar{C}) = \psi_i((T_r^1)_0, \bar{C}) = 0 = \phi_i^{\hat{o}}(N_0, C)$. It is sufficient to prove that $\psi_i(\{i,j\}_0, C) = \psi_j(\{i,j\}_0, C) = 0$ for the *mcstp* $(\{i,j\}_0, C)$ such that $c_{0j} = c_{ij} = 0$ and $c_{0i} = x$.

Since $m(\{i,j\}_0, C) = 0$, we assume that

$$\psi_i(\{i,j\}_0, C) = -\psi_j(\{i,j\}_0, C).$$

We prove that $\psi_j(\{i,j\}_0, C) = 0$.

As $|\mathcal{N}| \geq 3$, consider the *mcstp* $(\{i,j,k\}_0, C')$ such that $c'_{0i} = x$ and $c'_{hl} = 0$ otherwise. Since $m(\{i,j,k\}_0, C') = m(\{i,j\}_0, C') + m(\{k\}_0, C') = m(\{i,k\}_0, C') + m(\{j\}_0, C')$, by PM, $\psi_j(\{i,j,k\}_0, C') = \psi_j(\{i,j\}_0, C) = \psi_j(\{j\}_0, C) = 0$.

Case 2. $0 \notin T_r$. In this case, for each $i \in T_r$, $c_{0i} = x$.

We know that $\psi((T_r)_0, C) = \psi((T_r)_0, \bar{C}) = \psi((T_r)_0, x\widehat{C})$. By Claim 2, $\phi^{\hat{o}}((T_r)_0, C) = x\hat{o}(T_r) = x\psi((T_r)_0, \widehat{C})$. Then, to show that $\psi(N_0, C) = \phi^{\hat{o}}(N_0, C)$, we only need to prove that

$$\psi((T_r)_0, x\widehat{C}) = x\psi((T_r)_0, \widehat{C}), \text{ where } x \geq 0.$$

We distinguish two subcases:

Subcase 2.a. $x = \frac{p}{q}$ where $p, q \in \mathbb{N}$. Since ψ satisfies CA, it is straightforward that $\psi((T_r)_0, x\widehat{C}) = x\psi((T_r)_0, \widehat{C})$.

Subcase 2.b. $x \in \mathbb{R}^+ \setminus \mathbb{Q}^+$. There exists $\{x^p\}_{p \in \mathbb{N}}$ such that for each $p \in \mathbb{N}$, $0 < x^p < x$,

$x^p \in \mathbb{Q}^+$ and $\lim_{p\to\infty} x^p = x$. Thus, for each $p \in \mathbb{N}$ and each $i \in T_r$,

$$\psi_i((T_r)_0, x\widehat{C}) - x^p\psi_i((T_r)_0, \widehat{C}) \quad = \quad \psi_i((T_r)_0, x\widehat{C}) - \psi_i((T_r)_0, x^p\widehat{C}).$$

Since the *mcstp* $((T_r)_0, (x - x^p)\widehat{C})$ and $((T_r)_0, x^p\widehat{C})$ are similar,

$$\psi_i((T_r)_0, x\widehat{C}) - \psi_i((T_r)_0, x^p\widehat{C}) = \psi_i((T_r)_0, (x - x^p)\widehat{C}).$$

In addition, by PM

$$\sum_{j\in T_r\setminus\{i\}} \psi_j((T_r)_0, (x - x^p)\widehat{C}) \leq \sum_{j\in T_r\setminus\{i\}} \psi_j((T_r \setminus \{i\})_0, (x - x^p)\widehat{C}).$$

So, $0 \leq \psi_i((T_r)_0, (x - x^p)\widehat{C}) \leq (x - x^p)m((T_r)_0, \widehat{C}) = x - x^p$.
Therefore,

$$\begin{aligned} 0 \quad &\leq \quad \lim_{p\to\infty} [\psi_i((T_r)_0, x\widehat{C}) - x^p\psi_i((T_r)_0, \widehat{C})] \\ &= \quad \psi_i((T_r)_0, x\widehat{C}) - x\psi_i((T_r)_0, \widehat{C}) \\ &\leq \quad \lim_{p\to\infty} (x - x^p) = 0. \end{aligned}$$

Then, for each $i \in T_r$, $\psi_i((T_r)_0, x\widehat{C}) = x\psi_i((T_r)_0, \widehat{C})$.

■

■

According to Theorem 1, we have a characterization of the family of Kruskal sharing rules when $|\mathcal{N}| \geq 3$. Moreover, we have obtained an expression for the sharing function associated with a Kruskal sharing rule. For any set of possible agents $\mathcal{N}$, we have similar results if we add NN.

Theorem 2 *A rule ψ satisfies PM, CA, and NN if and only if*

$$\psi(N_0, C) = \phi^{\hat{o}}(N_0, C),$$

where $\hat{o}$ is the sharing function defined, for each $S \in 2^N \setminus \{\emptyset\}$, by

$$\hat{o}(S) = \psi(S_0, \widehat{C})$$

and the cost matrix $\widehat{C} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$.

Proof.

Existence.

By Proposition 1, Kruskal sharing rules satisfy PM, CA, and NN.

Uniqueness.

Consider a rule ψ satisfying PM, CA, and NN.

If $|\mathcal{N}| \geq 3$, we invoke Theorem 1 and, if $|\mathcal{N}| = 2$, we follow the procedure used in Theorem 1 except for the *mcstp* $(\{i,j\}_0, C)$ with $c_{0j} = c_{ij} = 0$ and $c_{0i} = x$. In this case, applying NN and considering that $\psi_i(\{i,j\}_0, C) + \psi_j(\{i,j\}_0, C) = 0$, we obtain that $\psi_i(\{i,j\}_0, C) = \psi_j(\{i,j\}_0, C) = 0$. ■

The properties stated in Theorem 2 are independent.

- The equal division rule, $\delta_i(N_0, C) = \frac{1}{n} m(N_0, C)$ for each $i \in N$, satisfies NN and CA. However, δ does not satisfy PM. Indeed, consider the *mcstp* (N_0, C), where $N = \{1, 2\}$ and $C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. In this case, $\delta_1(\{1\}_0, C) = 0$ while $\delta_1(N_0, C) = \frac{1}{2}$.

- Consider the subset of orders

$$\widetilde{\Pi}^N = \{\pi \in \Pi^N | \pi(i) < \pi(j) \text{ when } c_{0i} \leq c_{0j} \text{ for each } i, j \in N, i \neq j\}.$$

  Let β be the rule defined, for each $i \in N$, by

$$\beta_i(N_0, C) = \frac{1}{|\widetilde{\Pi}^N|} \sum_{\pi \in \widetilde{\Pi}^N} (v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi))).$$

  This rule satisfies PM and NN. However, it violates CA. Indeed, let $N = \{1, 2\}$ and consider the cost matrices

$$C = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} \text{ and } C' = \begin{pmatrix} 0 & 4 & 5 \\ 4 & 0 & 2 \\ 5 & 2 & 0 \end{pmatrix}.$$

The *msctp* (N_0, C) and (N_0, C') are similar, but $\beta_1(N_0, C+C') = 7 \neq \beta_1(N_0, C) + \beta_1(N_0, C') = 2 + 4 = 6$.

- Finally, consider the rule γ defined by

  1. If $\mathcal{N}$ has at least three members, for each $N \subset \mathcal{N}$, $\gamma(N_0, C) = Sh(N, v_{C^*})$.
  2. If $|\mathcal{N}| \leq 2$, for each $N \subset \mathcal{N}$, $\gamma(N_0, C) = Sh(N, v_C)$.

  Since the Shapley value satisfies additivity (Shapley, 1953a), the rule γ satisfies CA. For PM, we only need to prove that it is satisfied when $|\mathcal{N}| \leq 2$ because $Sh(N, v_{C^*})$ satisfies PM (Bergantiños and Vidal-Puga, 2007b).

  As the remaining cases are straightforward, we can assume that $N = \{i, j\}$ and $c_{0i} \leq c_{0j}$. We must prove that $\gamma_i(N_0, C) \leq c_{0i}$ and $\gamma_j(N_0, C) \leq c_{0j}$. We distinguish three cases:

  1. $c_{0i} \leq c_{ij} \leq c_{0j}$. We obtain that
     $\gamma_i(N_0, C) = c_{0i} + \frac{c_{ij} - c_{0j}}{2} \leq c_{0i}$ and $\gamma_j(N_0, C) = \frac{c_{ij} + c_{0j}}{2} \leq c_{0j}$.
  2. $c_{0i} \leq c_{0j} \leq c_{ij}$. In this case, $\gamma_i(N_0, C) = c_{0i}$ and $\gamma_j(N_0, C) = c_{0j}$.
  3. $c_{ij} \leq c_{0i} \leq c_{0j}$. We have that
     $\gamma_i(N_0, C) = c_{0i} + \frac{c_{ij} - c_{0j}}{2} \leq c_{0i}$ and $\gamma_j(N_0, C) = \frac{c_{ij} + c_{0j}}{2} \leq c_{0j}$.

  This rule fails NN. Consider (N_0, C) with $C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then, $\gamma(N_0, C) = (\frac{1}{2}, -\frac{1}{2})$.

Remark 1 *In view of Proposition 1 and Theorem 1, if $|\mathcal{N}| \geq 3$, a rule satisfying PM and CA also satisfies SCM and CON.*

Similarly, by Proposition 1 and Theorem 2, a rule satisfying PM, CA, and NN satisfies SCM and CON.

4 Kruskal sharing rules and cooperative games

In this section we study the relationship between weighted Shapley values for different TU games and Kruskal sharing rules.

A game with transferable utility, *TU game*, is a pair (N, v) where $N \subset \mathcal{N}$ and $v : 2^N \to \mathbb{R}$ satisfies that $v(\emptyset) = 0$.

A quite standard approach for defining rules in some problems is based on the theory of TU games. We first associate with each problem a TU game. In the case of *mcstp*, two games can be considered: the pessimistic game (Bird, 1976) and the optimistic game (Bergantiños and Vidal-Puga, 2007a).

- *The pessimistic game* associated with an *mcstp* (N_0, C) is denoted by (N, v_C). The value of each coalition $S \subset N$ is the cost of connecting agents in S to the source, assuming that agents in $N \setminus S$ are not present:

$$v_C(S) = m(S_0, C).$$

- *The optimistic game* associated with an *mcstp* (N_0, C) is denoted by (N, v_C^+). The value of each coalition $S \subset N$ is the cost of connecting agents in S to the source, assuming that agents in $N \setminus S$ are already connected, and agents in S can connect to the source through agents in $N \setminus S$:

$$v_C^+(S) = m(S_0, C^{+(N \setminus S)})$$

  where for each $i, j \in S$, $c_{ij}^{+(N \setminus S)} = c_{ij}$ and for each $i \in S$, $c_{0i}^{+(N \setminus S)} = \min_{j \in (N \setminus S)_0} c_{ji}$.

Given an *mcstp* (N_0, C), we can associate with it two additional TU games using its irreducible form: (N, v_{C^*}) and $(N, v_{C^*}^+)$.

Once the associated TU game has been chosen, we can compute a solution for TU games. Thus, the rule in the original problem is defined as the solution applied to the TU game associated with the original problem.

Given a family of TU games H, a *solution* on H is a function f which assigns to each TU game $(N, v) \in H$ the vector $(f_1(N, v), \ldots, f_n(N, v)) \in \mathbb{R}^N$, where the real number $f_i(N, v)$ is the payoff of $i \in N$ in the game (N, v) according to f. Several solutions have been defined for TU games. One of the best known solutions is the Shapley value.

The *Shapley value* (Shapley, 1953b) assigns to each TU game (N, v) the vector $Sh(N, v)$ where for each $i \in N$,

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi^N} \left[v(Pre(i, \pi) \cup \{i\}) - v(Pre(i, \pi)) \right].$$

In the literature on *mcstp*, several rules have been defined using solutions for an associated TU game. For instance, Kar (2002) studied the Shapley value of (N, v_C) whereas Bergantiños and Vidal-Puga (2007b) studied the Shapley value of (N, v_{C^*}).

Moreover, Bergantiños and Vidal-Puga (2007a) proved that $Sh(N, v^+_{C^*}) = Sh(N, v^+_C) = Sh(N, v_{C^*})$[1].

Shapley (1953a) introduced the family of weighted Shapley values for TU games. Each weighted Shapley value associates a payoff with each player according to a set of positive weights over the set of players. These weights are the proportions in which the players share in unanimity games. Kalai and Samet (1987) studied this family.

Given $N \subset \mathcal{N}$ and $w = \{w_i\}_{i \in N}$, we say that w is a *weight system* for N if for each $i \in N$, $w_i > 0$.

Take $N \subset \mathcal{N}$ and a weight system $w = \{w_i\}_{i \in N}$. The *weighted Shapley value* Sh^w associates with each TU game (N, v) a vector $Sh^w(N, v) \in \mathbb{R}^N$ such that for each $i \in N$,

$$Sh_i^w(N, v) = \sum_{\pi \in \Pi^N} p_w(\pi) \left[v(Pre(\pi, i) \cup \{i\}) - v(Pre(\pi, i)) \right]$$

where $p_w(\pi) = \prod_{j=1}^{n} \frac{w_{\pi^{-1}(j)}}{\sum_{k=1}^{j} w_{\pi^{-1}(k)}}$.

It is well-known that the Shapley value is a weighted Shapley value where for each $i, j \in N$, $w_i = w_j$.

Remark 2 *Kalai and Samet (1987) assume that the population of agents is fixed. Thus, they define the weight system with respect to N. Since we work with population monotonicity, we can not make this assumption. Hence, we define the weight system with respect to the set of possible agents $\mathcal{N}$.*

From now on, we say that $w = \{w_i\}_{i \in \mathcal{N}}$ is a *weight system* for $\mathcal{N}$ if for each $i \in \mathcal{N}$, $w_i > 0$. Given the weight system w and $N \subset \mathcal{N}$, we denote $w_N = \{w_i\}_{i \in N}$.

We now apply the *weight system* to the *mcstp* through the optimistic and pessimistic games.

- We say that ψ is an *optimistic weighted Shapley rule* for *mcstp* if there exists a

[1] Bergantiños and Vidal-Puga (2007a) proved that $v^+_{C^*} = v^+_C$. Therefore, $Sh^{w_N}(N, v^+_{C^*}) = Sh^{w_N}(N, v^+_C)$.

weight system $w = \{w_i\}_{i\in\mathcal{N}}$ such that for each *mcstp* (N_0, C),

$$\psi(N_0, C) = Sh^{w_N}(N, v^{+}_{C^*})$$

- We say that ψ is a *pessimistic weighted Shapley rule* for *mcstp* if there exists a weight system $w = \{w_i\}_{i\in\mathcal{N}}$ such that for each *mcstp* (N_0, C),

$$\psi(N_0, C) = Sh^{w_N}(N, v_{C^*}).$$

Bergantiños and Lorenzo-Freire (2008) proved that the optimistic weighted Shapley rules are Kruskal sharing rules where the sharing function for an agent i in a coalition S is proportional to his weight, *i.e.*, for each $S \in 2^N\backslash\{\emptyset\}$ and each $i \in S$, $o_i^{w_N}(S) = \frac{w_i}{\sum_{j\in S} w_j}$. These authors also define the pessimistic weighted Shapley rules, proving that the families of optimistic and pessimistic weighted Shapley rules are different. However, they do not ask whether the pessimistic weighted Shapley rules are Kruskal sharing rules or not. In this paper we study the relationship between both families, proving that the pessimistic weighted Shapley rules are also Kruskal sharing rules.

In accordance with Theorem 2, it is easy to calculate the sharing function for any Kruskal sharing rule. Then, we will apply this theorem not only to show that the pessimistic weighted Shapley rules are Kruskal sharing rules, but also to calculate the associated sharing function. The same procedure could be applied in the case of the optimistic weighted Shapley rules, obtaining the same result as Bergantiños and Lorenzo-Freire (2008), but following a completely different proof.

Corollary 1 *Let φ^w be the pessimistic weighted Shapley rule associated with the weight system w. Thus, for each mcstp (N_0, C),*

$$\varphi^w(N_0, C) = \phi^{o^{w_N}}(N_0, C),$$

where the sharing function o^{w_N} is given, for each $S \in 2^N \setminus \{\emptyset\}$ and each $i \in S$, by

$$o_i^{w_N}(S) = \sum_{\pi\in\Pi(S\backslash\{i\})} \prod_{j=1}^{s-1} \frac{\omega_{\pi^{-1}(j)}}{\sum_{k=1}^{j} \omega_{\pi^{-1}(k)} + \omega_i}.$$

Proof.

Since Bergantiños and Vidal-Puga (2007a) proved that

$$v_{C^*}(Pre(i,\pi)\cup\{i\}) - v_{C^*}(Pre(i,\pi)) = \min_{j\in Pre(i,\pi)_0}\{c^*_{ij}\} \geq 0,$$

φ^w satisfies NN.

The weighted Shapley value satisfies additivity (Kalai and Samet, 1987). Moreover, $v_{(C+C')^*} = v_{C^*} + v_{C'^*}$, where (N_0, C) and (N_0, C') are two similar *mcstp* (Bergantiños and Vidal-Puga, 2008).

Using these results, for each weight system w

$$\begin{aligned}
\varphi^w(N_0, C + C') &= Sh^{w_N}(N, v_{(C+C')^*}) \\
&= Sh^{w_N}(N, v_{C^*} + v_{C'^*}) \\
&= Sh^{w_N}(N, v_{C^*}) + Sh^{w_N}(N, v_{C'^*}) \\
&= \varphi^w(N_0, C) + \varphi^w(N_0, C').
\end{aligned}$$

Then, φ^w satisfies CA.

Let us denote $N^{-j} = N \setminus \{j\}$. To prove that the weighted Shapley value satisfies PM, we need to prove that for each $i \in N^{-j}$

$$\begin{aligned}
& Sh_i^{w_N}(N, v_{C^*}) = \sum_{\pi\in\Pi^N} p_{w_N}(\pi)\left[v_{C^*}(Pre(i,\pi)\cup\{i\}) - v_{C^*}(Pre(i,\pi))\right] \\
\leq & \sum_{\pi^{-j}\in\Pi(N^{-j})} p_{w_{N^{-j}}}(\pi^{-j})\left[v_{C^*}(Pre(i,\pi^{-j})\cup\{i\}) - v_{C^*}(Pre(i,\pi^{-j}))\right] \\
= & \; Sh_i^{w_{N^{-j}}}(N^{-j}, v_{C^*}).
\end{aligned}$$

We know that $\sum\limits_{\pi\in\Pi^N} p_{w_N}(\pi)\left[v_{C^*}(Pre(i,\pi)\cup\{i\}) - v_{C^*}(Pre(i,\pi))\right]$

$$\begin{aligned}
= & \sum_{\pi\in\Pi^N, j\in Pre(i,\pi)} p_{w_N}(\pi) \min_{k\in Pre(i,\pi)_0}\{c^*_{ik}\} + \sum_{\pi\in\Pi^N, j\notin Pre(i,\pi)} p_{w_N}(\pi) \min_{k\in Pre(i,\pi)_0}\{c^*_{ik}\} \\
\leq & \sum_{\pi\in\Pi^N, j\in Pre(i,\pi)} p_{w_N}(\pi) \min_{k\in (Pre(i,\pi)\setminus\{j\})_0}\{c^*_{ik}\} \\
+ & \sum_{\pi\in\Pi^N, j\notin Pre(i,\pi)} p_{w_N}(\pi) \min_{k\in Pre(i,\pi)_0}\{c^*_{ik}\}.
\end{aligned}$$

Given a cost matrix C, we know that $C^* \leq C$. Considering the connection costs of agents in N_0^{-j}, $(C^*)^{-j} \leq C^{-j}$, where C^{-j} denotes the restriction of C to the agents in N^{-j}. Moreover, $(C^*)^{-j}$ is an irreducible matrix (Bergantiños and Vidal-Puga, 2007b). Thus, $(C^*)^{-j} \leq (C^{-j})^*$.

On the other hand, we denote $\pi_{N^{-j}}$ as the restriction of π to N^{-j}

Then, for each $i \in N^{-j}$

$$\begin{aligned} Sh_i^{w_N}(N, v_{C^*}) &\leq \sum_{\pi \in \Pi^N, j \in Pre(i,\pi)} p_{w_N}(\pi) \min_{k \in (Pre(i,\pi) \setminus \{j\})_0} \{(c^{-j})^*_{ik}\} \\ &+ \sum_{\pi \in \Pi^N, j \notin Pre(i,\pi)} p_{w_N}(\pi) \min_{k \in Pre(i,\pi)_0} \{(c^{-j})^*_{ik}\} \\ &= \sum_{\pi^{-j} \in \Pi(N^{-j})} \left\{ \sum_{\pi \in \Pi^N, \pi_{N^{-j}} = \pi^{-j}} p_{w_N}(\pi) \min_{k \in Pre(i,\pi^{-j})_0} \{(c^{-j})^*_{ik}\} \right\}. \end{aligned}$$

Bergantiños and Lorenzo-Freire (2008) prove that

$$\sum_{\pi \in \Pi^N, \pi_{N^{-j}} = \pi^{-j}} p_{w_N}(\pi) = p_{w_{N^{-j}}}(\pi^{-j}).$$

Thus, for each $i \in N^{-j}$

$$\begin{aligned} Sh_i^{w_N}(N, v_{C^*}) &\leq \sum_{\pi^{-j} \in \Pi(N^{-j})} p_{w_{N^{-j}}}(\pi^{-j}) \left\{ \min_{k \in Pre(i,\pi^{-j})_0} \{(c^{-j})^*_{ik}\} \right\} \\ &= Sh_i^{w_{N^{-j}}}(N^{-j}, v_{C^*}). \end{aligned}$$

Then, by Theorem 2, these rules are Kruskal sharing rules. To obtain the corresponding sharing function, we consider an *mcstp* (N_0, C) and a weight system w.

The sharing function is, for each $S \in 2^N \setminus \{\emptyset\}$ and each $i \in S$,

$$
\begin{aligned}
o_i^{w_N}(S) &= \varphi_i^w(S_0,\widehat{C}) \\
&= \sum_{\pi\in\Pi(S)} p_{w_S}(\pi)\left[v_{\widehat{C}}(Pre(i,\pi)\cup\{i\}) - v_{\widehat{C}}(Pre(i,\pi))\right] \\
&= \sum_{\pi\in\Pi(S)} p_{w_S}(\pi) \min_{k\in Pre(i,\pi)_0}\{\widehat{c}_{ik}\} \\
&= \sum_{\pi\in\Pi(S):\pi(i)\neq 1} p_{w_S}(\pi)0 + \sum_{\pi\in\Pi(S):\pi(i)=1} p_{w_S}(\pi)1 \\
&= \sum_{\pi\in\Pi(S):\pi(i)=1} p_{w_S}(\pi) = \sum_{\pi\in\Pi(S\setminus\{i\})} \prod_{j=1}^{s-1} \frac{w_{\pi^{-1}(j)}}{\sum\limits_{k=1}^{j} w_{\pi^{-1}(k)} + w_i}.
\end{aligned}
$$

■

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